

Thermoelastic Analysis of Non-Local Infinitely Solid Cylinders Composed of Two Different Materials

A. I. Hassanien, M. H. E. Mohamed, S. M. Ali

1. Department of Mathematics, Faculty of Science, Ain Shams University, Egypt

Abstract

In this work, we consider a two-dimensional problem of an infinitely long solid cylinder consisting of two different homogeneous and isotropic thermoelastic materials within the context of the fractional order theory of thermoelasticity. The lateral surface of the cylinder is taken to be traction free and is subjected to a known temperature distribution which is a function of time t and z . There are no body forces or heat sources affecting the medium. Laplace and exponential Fourier transform techniques are used to solving the problem. The inverse Laplace and exponential Fourier transforms are obtained using a numerical technique.

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The predictions of the fractional order theory are discussed and compared with those for the generalized theory of thermoelasticity. We also study the effect of the fractional derivative parameters of the two media on the behavior of the solution. Numerical results are computed and represented graphically for the temperature, displacement and stress distributions.

Key words: Fractional calculus; Solid Cylinder; Different Materials; Laplace transform; Fourier transforms

1. INTRODUCTION

Lord and Shulman [1] introduced the theory of generalized thermoelasticity with one relaxation time by using the Maxwell-Cattaneo law of heat conduction instead of the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity. The uniqueness of solution for this theory was proved under different conditions by Ignaczak [2], Sherief and Dhaliwal [3] and by Sherief [4]. Exact solution for a problem of a spherical cavity was obtained by Sherief and Saleh [5]. Some problems for a penny-shaped crack and a mode I crack were solved by Sherief and El-Maghraby [6-7]. This theory was extended by Sherief et al. [8] to micropolar media. Anwar and Sherief [9] studied A Problem in Generalized Thermoelasticity for an Infinitely Long Annular Cylinder Composed of Two Different Materials.

Fractional calculus has been used successfully to modify many existing models of physical processes. Caputo and Mainardi [10-11] and Caputo [12] found good agreement with experimental results when using fractional derivatives for a description of viscoelastic materials and established the connection between fractional derivatives and the theory of linear viscoelasticity.

The solution obtained by using ordinary derivatives predicts an instantaneous response while that obtained by using fractional derivatives predicts a retarded response that depends on the history of the applied causes. This is more in accord with physical observations [13].

The general space-time-fractional heat conduction equation in the one-dimensional case has been formulated by Gorenflo et al [14]. Povstenko [15] made a review of thermoelasticity that uses fractional heat conduction equation. The theory of thermal stresses based on the heat conduction equation with the Caputo time-fractional derivative is used by Povstenko [16] to investigate thermal stresses in an infinite body with a



circular cylindrical hole. Povstenko proposed and investigated new models that use fractional derivative in [17-18].

The fractional order theory of thermoelasticity was derived by Sherief et al. [19]. It is a generalization of both the coupled and the generalized theories of thermoelasticity. Sherief and Abd El Latief [20, 21] have solved a 1D problems

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for a half space and for spherical cavity in this theory, solved a 2D problem of half- space [22], studied the effect of the fractional derivative parameter on fractional thermoelastic material with variable thermal conductivity [23] and applied this theory to a 1D problem for a half-space overlaid by a thick layer of a different materials [24].

Recently, Hamza et al. [25] derive a new theory of thermoelasticity associated with two relaxation times using the methodology of fractional calculus, Hamza et al. [26] have solved 1D problems in the context of this theory, derive a new mathematical model of Maxwell's equations in an electromagnetic field using the physical principles of fractional calculus[27]. Some contribution works that use fractional calculus can be found in [28-30].

In this work, we solve a 2D problem of an infinitely long solid cylinder consisting of two different homogeneous and isotropic thermo elastic materials within the context of the fractional order theory of thermoelasticity [19]. The lateral surface of the cylinder is taken to be traction free and is subjected to a known temperature distribution. The solution is obtained for different values of the fraction parameters of the two media. The fractional parameters effects on the media in radial and axial directions are discussed.

2. FORMULATION OF THE PROBLEM

We consider a two-dimensional problem of an infinitely long solid cylinder consisting of two different homogeneous and isotropic thermo elastic materials. The inner layer occupies the region $0 \leq r \leq a, -\infty < z < \infty$ and the outer layer occupies the region $a \leq r \leq b, -\infty < z < \infty$, where (r, ϕ, z) are cylindrical polar coordinates with unit vectors e_r, e_ϕ and e_z . The lateral surface of the cylinder is taken to be traction free and is subjected to a known temperature distribution which is a function of time t and z . There are no body forces or heat sources affecting the medium. Due to the physical of the problem, all considered quantities will depend on r, z and t only. The displacement vector u_i takes the form

$u_i = u_i(r, z, t)e_r + w_i(r, z, t)e_z, i = 1, 2.$ (1) where the suffix 1 refers to the inner medium and the suffix 2 refers to the outer medium. The cubical dilatation e_i in medium i is thus given by

$$e_i = \frac{1}{r} \left(\frac{\partial}{\partial r} (ru_i) + \frac{\partial}{\partial z} (rw_i) \right) \quad (2)$$

The equation of motion in vector form can be written as

$$\rho_i \frac{\partial^2 u_i}{\partial t^2} = (\lambda_i + \mu_i) \text{grad div } u_i - \lambda_i \text{ grad } T_i + \mu_i \nabla^2 u_i \quad (3)$$

where λ_i, μ_i are Lamé's constants, ρ_i is the density, T_i is the absolute temperature and α_i is a material constant given by $(3\lambda_i + 2\mu_i)\alpha_i$ where α_i is the coefficient of linear thermal expansion. ∇^2 is Laplace's operator in cylindrical polar coordinates, given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$

$$(r^*, z^*, u, w^*, \theta, \tau^*) \in C_{11}(\Omega \times \Omega \times \Omega \times \Omega \times \Omega \times \Omega), \quad (r, z, u, w, \theta, \tau) \in C_{11}(\Omega \times \Omega \times \Omega \times \Omega \times \Omega \times \Omega)$$

$$\sigma_{ij}^* = \frac{\sigma_{ij}}{\lambda_1 + 2\mu_1} \theta, \quad \gamma_1 \frac{T_i - T_0}{k c_{1111}} (q_{ri}^*, q_{zi}^*) \in C_{11}(\Omega \times \Omega \times \Omega \times \Omega \times \Omega \times \Omega)$$

where $c_1 = \frac{\lambda_1 + 2\mu_1}{k_1}, \eta_1 = \frac{\rho_1 c E_1}{\rho_1}$

Using the above non-dimensional variables, (dropping the asterisks for convenience), The governing equations (3)-(7a) in non-dimensional form become

$$\nabla_i \cdot (\nabla_i u) - \nabla_i \nabla_i u = \frac{1}{t} \quad (8)$$

$$\nabla_i \nabla_i u = \frac{1}{t} \quad (9)$$

$$\nabla_i \nabla_i u = \frac{1}{t} \quad (10a)$$

$$\nabla_i \nabla_i u = \frac{1}{t} \quad (10b)$$

$$\nabla_i \nabla_i u$$

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$$\nabla_i \nabla_i u$$

$$\nabla_i \nabla_i u$$

$$\nabla_i \nabla_i u = \frac{1}{t} \quad (12a)$$

$$\nabla_i \nabla_i u = \frac{1}{t} \quad (12b)$$

$$\sigma_{zz} = \frac{1}{2} \left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial r} \right) \quad (12c)$$

$$\sigma_{rz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) \quad (12d)$$

$$\sigma_{r\theta} = \sigma_{z\theta} = \sigma_{\theta\theta} = 0, \text{ where } \sigma_{r\theta} = \tau_{r\theta}, \sigma_{z\theta} = \tau_{z\theta}, \sigma_{\theta\theta} = \tau_{\theta\theta} \quad (12e)$$

$$\sigma_{rr} = 0, \sigma_{rz} = 0, \sigma_{zz} = f(t,z), \text{ at } r = b \text{ where } b \text{ is the radius} \quad (13)$$

of the lateral cylinder.

The continuity conditions of the problem are given by

$$u_1 = u_2, w_1 = w_2, \sigma_{r1} = \sigma_{r2}, \sigma_{z1} = \sigma_{z2}, q_{r1} = q_{r2}, \text{ at } r = a. \text{ where } a \text{ is the radius of the} \quad (14)$$

inner cylinder.

3. SOLUTION IN THE TRANSFORMED DOMAIN

Applying the Laplace transform with parameter s defined by the relation

$$f(r, z, s) = \int_0^\infty f(r, z, t) e^{-st} dt$$

to both sides of equations (8-11), we obtain

$$\sigma_{rr} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \sigma_{r\theta} \right) + \text{grad}_i^2 V s_i^2 u_i \quad (15)$$

$$\sigma_{zz} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \sigma_{rz} \right) + \text{grad}_i^2 V s_i^2 u_i$$

$$\dots \frac{1}{r^2} \dots \frac{1}{r} \dots m_{22} \dots f_{*2} \dots f_{j2,1} \dots A_{2,j} I_0(\dots r)(k_{22j} \dots V S_2) \dots \quad (44)$$

$$\dots B_{2,j} K_0(\dots r)(k_{22j} \dots V S_2) \dots$$

The solutions of equations (43) and (44) take the form

$$\dots^* \dots f_1 \dots C I_1(m r_1) \dots A I_{11}(\dots r) \dots A I_{12}(\dots r) \dots \quad (45)$$

$$\dots^* \dots f_2 \dots C I_2(m r_2) \dots D K_2(m r_2) \dots \dots (A I_{2j}(\dots r) \dots B K_{2j} \dots (\dots r)) \dots \quad (46)$$

Thus the solution of equation (38) takes the form

$$\dots^* \dots u_1 \dots C m I m r_{111}(1) \dots A q I_{11j} \dots j \dots (\dots r) \dots \quad (47)$$

$$\dots \square_{2w} \dots e \dots \square \square 1 \square \dots \square \dots \square$$

$$u_{2j} = C_m I_m r_{221}^{(2)} D_m K_m r_{221}^{(2)} \int_{j-1}^j (A q I_{2j2j} (r_{2j}) + B q K_{2j2j} (r_{2j})) \quad (48)$$

In order to find the displacement components w_i applying Laplace and Fourier transforms to both sides of equation

(2)

$$\frac{\partial}{\partial r} \left(\frac{\partial w_i}{\partial r} \right) + \dots = \dots \quad (49)$$

Using the equations (29-30) and (47-48), we obtain

$$w_{1i} = \dots C_m I_{1102} (m r_1) \cdot p^2 \cdot A I_{10j} \dots \quad (50)$$

$$w_{2i} = \dots C_m I_m r_{22}^2 \cdot D_m K_m r_{20}^{(2)} \cdot p \dots \quad (51)$$

Applying Laplace and Fourier transforms into the equations (12), we obtain

$$\frac{\partial}{\partial r} \left(\frac{\partial u_i}{\partial r} \right) + \dots = \dots \quad (52a)$$

$$\frac{\partial}{\partial z} \left(\frac{\partial w_i}{\partial z} \right) + \dots = \dots \quad (52b)$$

$$\frac{\partial}{\partial r} \left(\frac{\partial w_i}{\partial r} \right) + \dots = \dots \quad (52c)$$

$$\frac{\partial}{\partial r} \left(\frac{\partial u_i}{\partial r} \right) + \dots = \dots \quad (52d)$$

Substituting from equations (27-30), (47-48) and (50-51) into equations (52), we get

$$\frac{\partial}{\partial r} \left(\frac{\partial u_i}{\partial r} \right) + \dots = \dots \quad (53)$$

a=1	$\mu_1 = 3.86 \times 10^{10}$	$T_0 = 293$	$\rho_1 = 2707$
b = 2	$\mu_2 = 3.88 \times 10^{10}$	$\tau_1 = 0.02$	$\rho_2 = 7144$
c ₁ =896	k ₁ =204	$\tau_2 = 0.005$	h = 0.2
c ₂ = 384.3	k ₂ = 112.2	$\lambda_1 = 4.7 \times 10^{10}$	$\lambda_2 = 9.07 \times 10^{10}$
$\alpha_{t1} = 8.418d-5$	$\alpha_{t2} = 4.106d-5$		

The computations were carried out for the function

$$f(r,z,t) = \begin{cases} 1 & \text{if } 0 \leq z \leq h \\ 0 & \text{otherwise} \end{cases}$$

$$f(r,z,t) = 0, \text{ at } r = b.$$

where h is a width of temperature distribution, applying Laplace and Fourier transforms, we obtain

$$f(r,z,t) = \frac{2 \sin ph}{ps} \dots$$

The investigation of the effect of the outer cylinder whose fractional derivative parameter α_2 at $z = 0$ in the radial direction has been carried out in the preceding discussions.

The computations were performed for a wide range of $(0 \leq r \leq 2)$, different values of $\alpha_2 \in [0.5, 0.999, 1]$, one value of time, namely $t = 0.13$ and $\alpha_1 = 1$. This enables us to represent the typical numerical results in Figures (1) - (3), for the temperature θ , displacement u and the thermal stresses σ_{rr} , respectively. These figures show that for $\alpha_2 \in [0.999, 1]$ the waves not pass to the inner media and it will vanish in the outer media whose fractional parameter $\alpha_1 = 1$. This result is in agreement with the generalized theory of thermoelasticity that the waves have finite speed. While $\alpha_2 \in [0.5, 0.999]$ the waves will pass to the inner media since it has infinite speed. This result is consistent with [24].

The investigation of the effect of the outer cylinder whose fractional derivative parameter α_2 inside the outer cylinder

$r = 1.5$ in the direction parallel to the axis of the cylinders has been carried out in the preceding discussions.

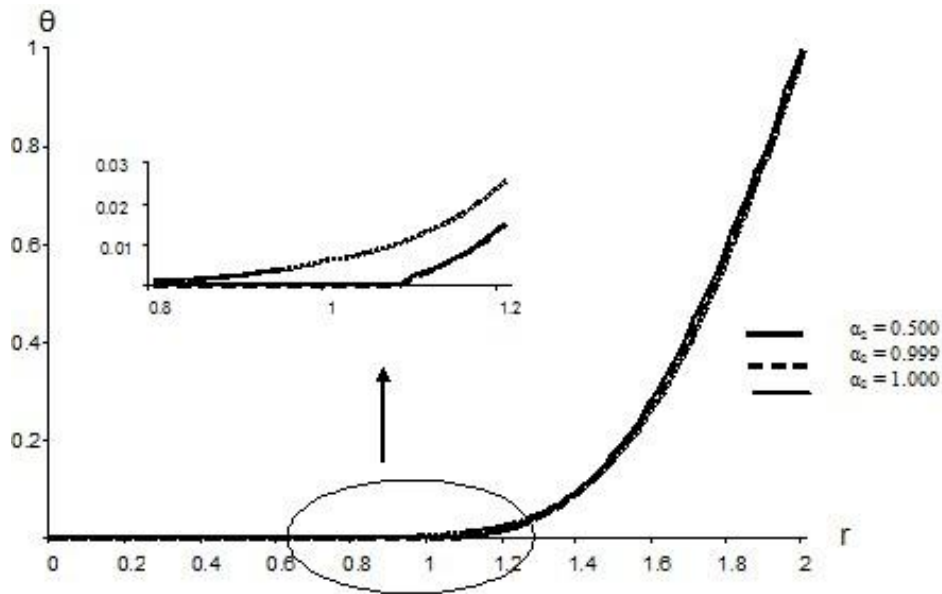


Fig. 1 α_2 radial effect on the temperature distribution for $\alpha_1=1$ and $t = 0.13$

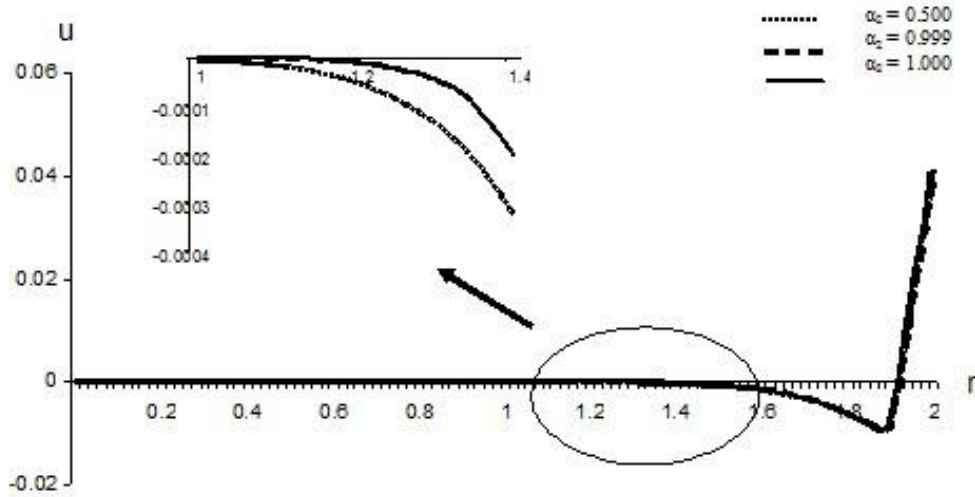


Fig. 2 α_2 radial effect on the displacement distribution for $\alpha_1=1$ and $t = 0.13$

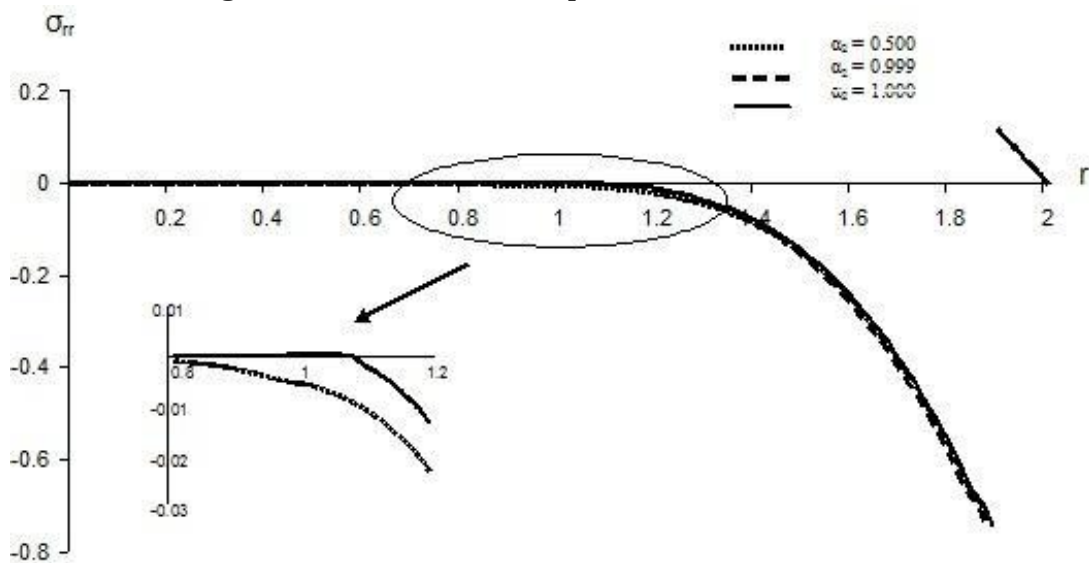


Fig. 3 α_2 radial effect on the stresses distribution for $\alpha_1=1$ and $t = 0.13$

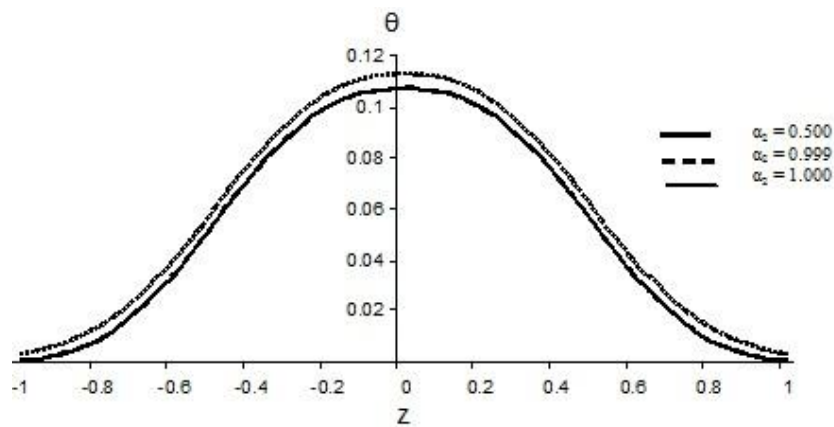


Fig. 4 α_2 axial effect on the temperature Distribution for $\alpha_1=1$ and $t = 0.1$

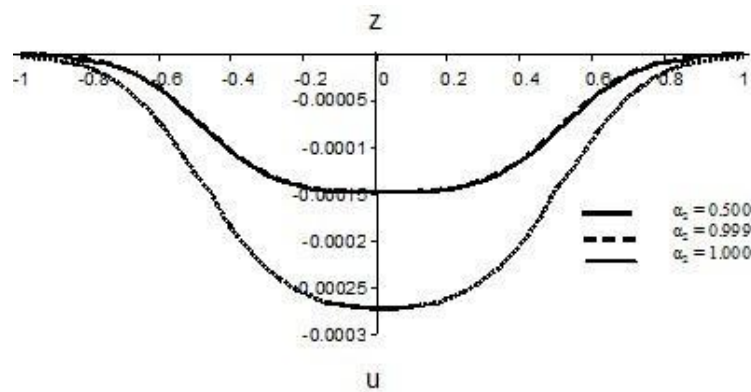


Fig. 5 α_2 axial effect on the displacement distribution for $\alpha_1=1$ and $t = 0.1$

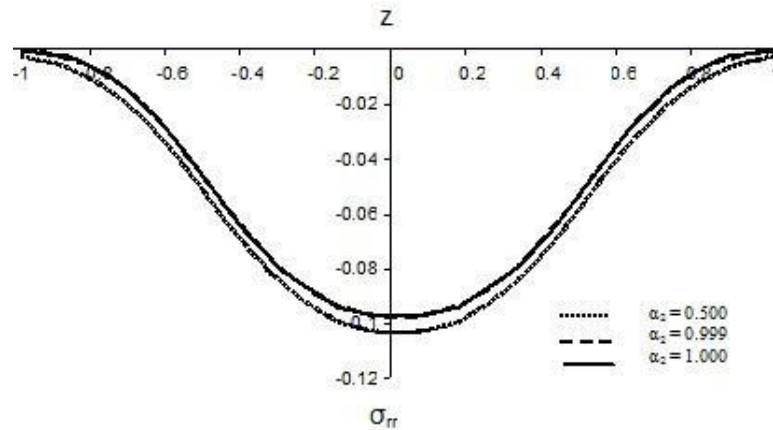


Fig. 6 α_2 axial effect on the stresses distribution for $\alpha_1=1$ and $t = 0.1$

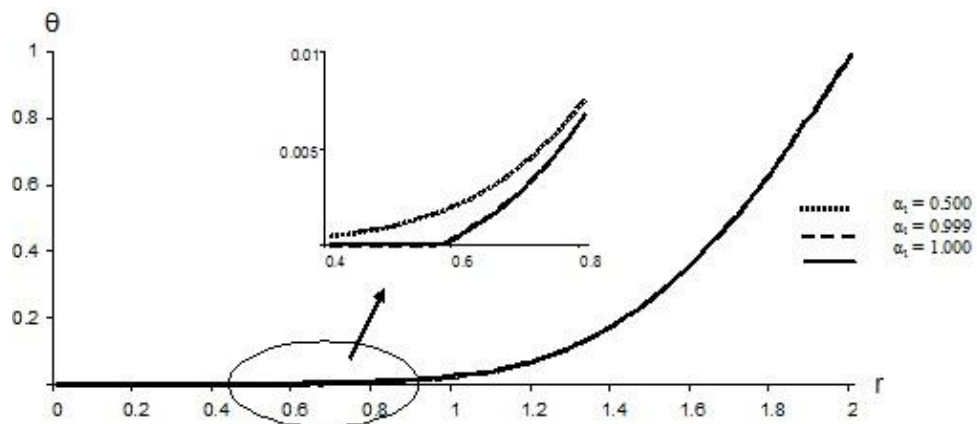


Fig. 7 α_1 radial effect on the temperature distribution for $\alpha_2=1$ and $t = 0.2$

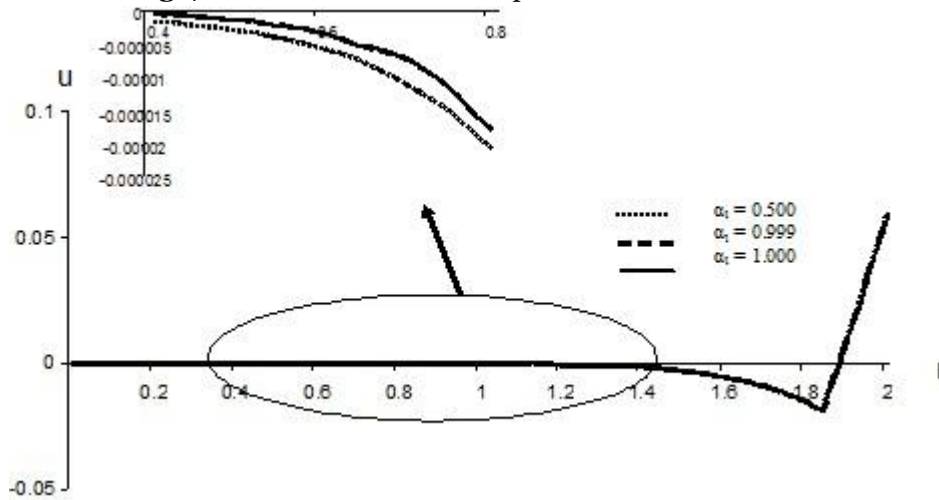


Fig. 8 α_1 radial effect on the displacement distribution for $\alpha_2=1$ and $t = 0.2$

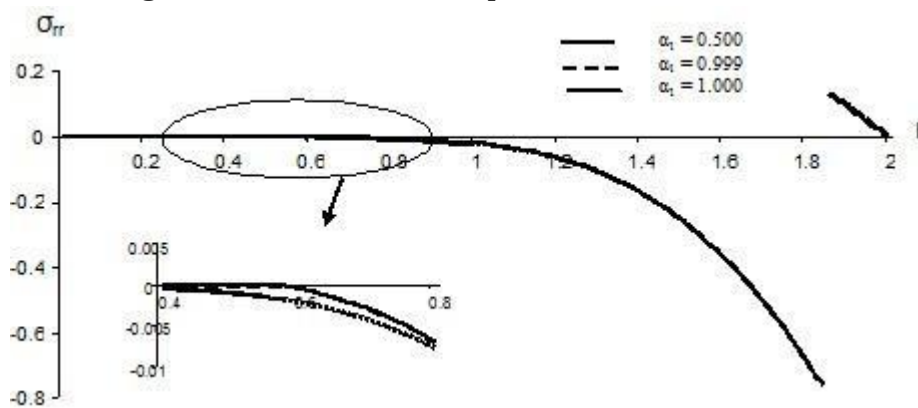


Fig. 9 α_1 radial effect on the stresses Distribution for $\alpha_2=1$ and $t = 0.2$

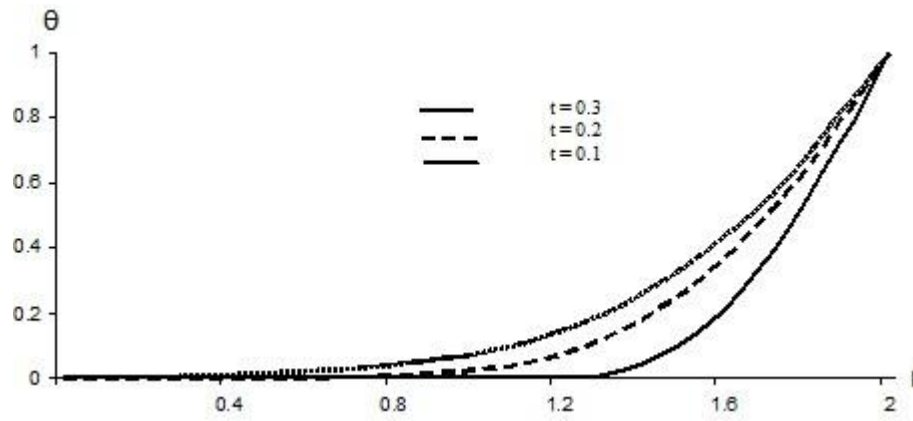


Fig. 10 α_1 radial effect on the temperature distribution for $\alpha_1=1$ and $\alpha_2=1$

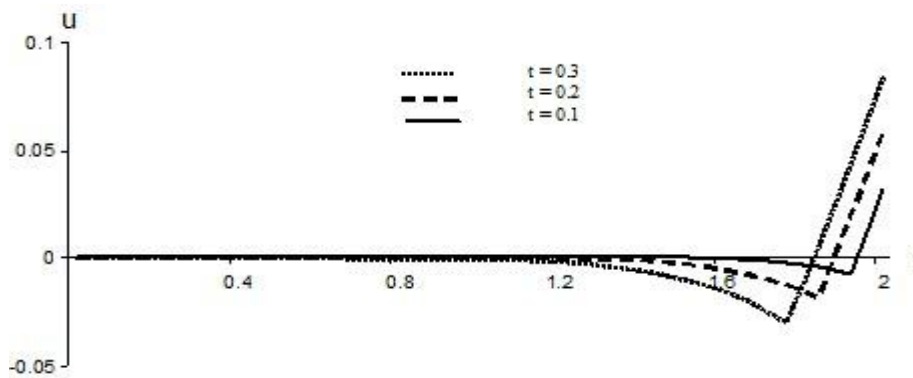


Fig. 11 Displacement Distribution for $\alpha_1=1$ and $\alpha_2=1$

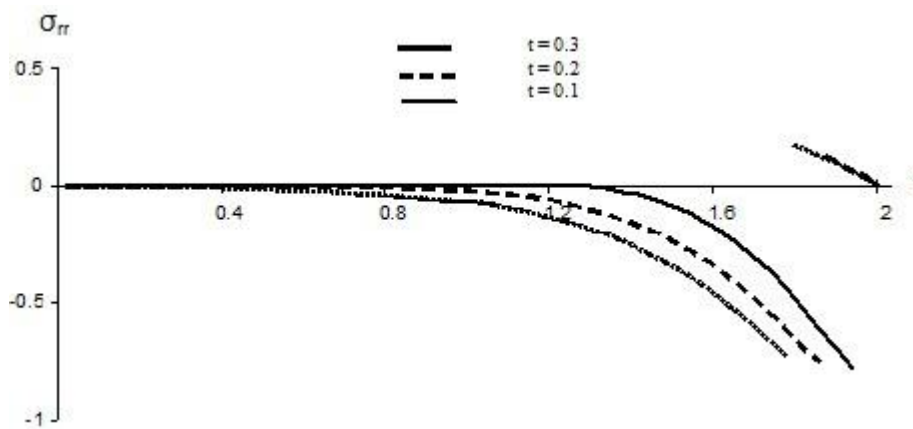


Fig. 12 Stresses Distribution for $\alpha_1=1$ and $\alpha_2=1$

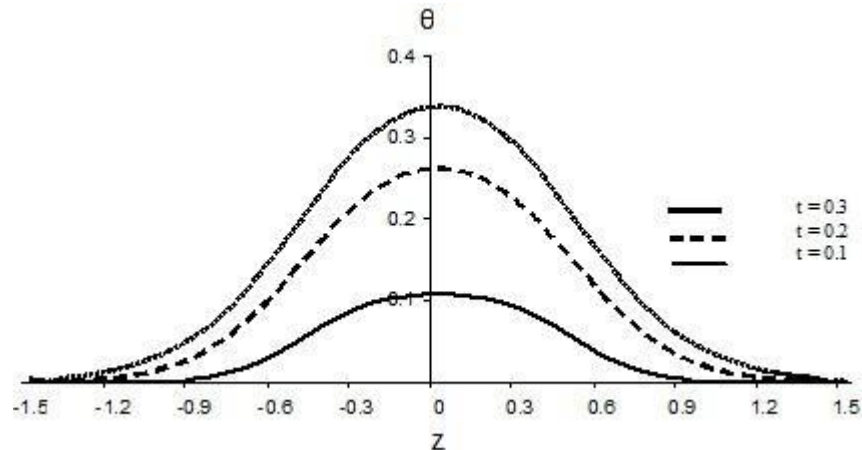


Fig. 13 Temperature Distribution for $\alpha_1=1$ and $\alpha_2=1$

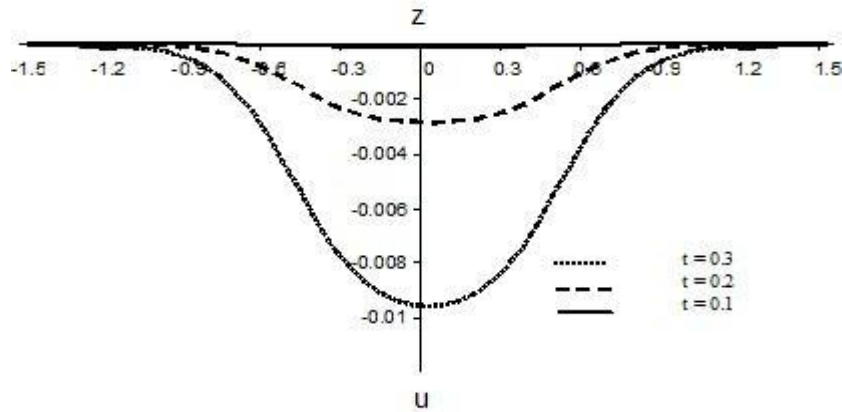


Fig. 14 Displacement Distribution for $\alpha_1=1$ and $\alpha_2=1$

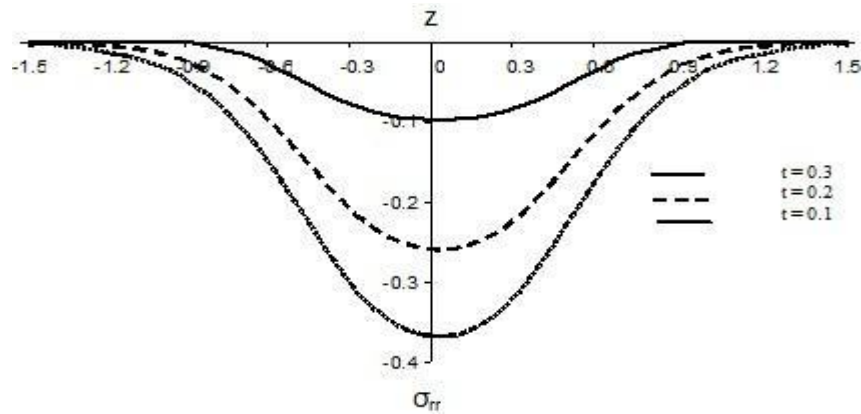


Fig. 15 Stresses Distribution for $\alpha_1=1$ and $\alpha_2=1$

Figures (4), (5) and (6), respectively, exhibit the temperature, displacement and stress distributions for a wide range of (

$\alpha_1 \in [0, 1]$, $\alpha_2 = \{0, 0.999, 1\}$ when $t = 0.1$, $\alpha_1 = 1$ at $r=1.8$ parallel to z -axis. Obviously, the effects of the fractional parameter α_2 on the field profiles in the axis direction are noticeable than the radial direction. Also for $\alpha_2 = 0.999, 1$ the waves vanish rapidly than when $\alpha_2 = 0.5$. Thus the wave has the finite speed for $\alpha_2 = 0.999, 1$ while for $\alpha_2 = 0.5$ it has infinite.

The investigation of the effect of the inner cylinder whose fractional derivative parameter α_1 at $z = 0$ in the radial direction has been carried out as follow.

Figures (7), (8) and (9), respectively, depict the temperature, displacement and stress distributions for $\alpha_1 = \{0, 0.999, 1\}$,

$\alpha_2 = 1$ when the time is large enough to penetrate to the inside media, namely, $t = 0.2$. These figures show the effects of the inner media fractional parameter α_1 . We notice that for $\alpha_1 = 0.999, 1$ the waves have finite speed while for $\alpha_1 = 0.5$ the wave has infinite speed.

Finally, the computation was carried out for different values of time, namely, $t = \{0.1, 0.2, 0.4\}$. The temperature, displacement and stress profiles are shown at $z = 0$ and $r = 1.5$ in figures (10-12) and (13-15), on respectively. These figures show that as the time increase as the waves penetrates the media to a larger distance.

5. CONCLUSION

When the fractional time parameter is close to one, then the solution seems to behave like the generalized theory of thermoelasticity. This result indicates that the fractional model of thermoelasticity may preserve the advantage of the generalized theory of thermoelasticity that the velocity of waves is finite.

The fractional parameters have a noticeable effect in the axial direction of the cylinders more than the radial direction.

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