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Efficient Iterative Methods for Matrix Equations: A Symmetric Arrowhead Solution Approach

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Abstract—Based on the conjugate gradient (CG) algorithm, the constrained matrix equation *AXB=C* and the associate optimal approximation problem are considered for the symmetric arrowhead matrix solutions in the premise of consistency. The convergence results of the method are presented. At last, a numerical example is given to illustrate the efficiency of this method.

I. INTRODUCTION

ET $R^{m\,n□}$ be the set of $m\,n□$ real matrices, $SAR^{n\,n□}$ be the set of $n\,n□$ real symmetric arrowhead matrices and I_n be the identity matrix of order n. For any $A□R^{m\,n□}$, A^T, $A^{\dagger}$, $\|A\|_F$ and

A denote the transpose, Moore-Penrose generalized inverse,

Frobenius norm and Euclid norm, respectively.

For any $AB, □\mathbf{R}^{m\,n□}$, $\langle A\ B\rangle$, □ *trace* $B\ A□\ {}^{T}□\ □\ 0$ denotes the inner product of A and B. Therefore, $R^{m\,n□}$ is a complete inner product space endowed with $\|A\|^2 □ \langle AA\rangle$, . For any matrices $A\ A_1,\ {}_2,□,\ A_k\ □\mathbf{R}^{m\ n□}$, if

□ *trace* $A□\ {}_i{}^T A_j□\ □\ 0□i□j□$, then it is easy to verify

${}_2,□, A_k$ are linearly independent and orthogonal.

Proposition 1. Let $A\ B,\ □\mathbf{R}^{n\ n□}$, then *trace* $A(\ )$ □ *trace* $A(\ {}^T)$; *trace* $AB(\ )$ □ *trace* $BA(\ )$ *trace* $A(\ )$□*trace* $B(\ )$

Definition 1. If a matrix $A\ □\ □a_{ij}\ □\ □\mathbf{R}^{n\,n□}$ satisfies the following

$$□a_{11}\,a_{12}\,a_{13}\ □\ a_{1n}\ □$$

$$□□a_{21}\,a_{22}\ 0\ □\ 0\ □□$$

$$A\ a□□\ {}_{31}\ 0\ a_{33}\ □\ 0\ □$$

$$□ \qquad □$$

$$□□\ □\ □\ □\ □□$$

$$□□a_{n1}\ 0 \quad 0\ □\ a_{nn}\ □□$$

then we denote that A is arrowhead matrix, this type of matrix

set is denoted as $\mathbf{AR}^{n\ n□}$. If $a_{i1}\ □\ a_{1i}\ (i\ □1,2,□,\ )n$, then we denote that A is symmetric arrow-head matrix, this type of matrix set is denoted as $\mathbf{SAR}^{n\ n□}$. *vec* A_i □ □ denotes a vector

$□a\ a_{11},\ {}_{21},□,a\ a\ a_{n1},\ {}_{22},\ {}_{33},□,a_{nn}\ □_T$.

Symmetric arrowhead matrices have many applications in the modern control theory which can represent the parameter matrix of nonlinear control systems or the large sparse matrix in the linear systems [1], [5]-[7]. With the development of electromagnetic compatibility, the mathematical representation of the influence factors of electromagnetic interference also has potential application value. In this paper, we consider the following constrained matrix equation

$$AXB\ C□ \tag{1}$$

in which $A□\mathbf{R}^{m\ n□}$, $B□\mathbf{R}^{n\ s□}$, $C\ □\mathbf{R}^{m\ s□}$. The above matrix equation and other constrained matrix equations have been studied in [2], [3], [8], [9], etc. Peng et al. [4] analyzed CG algorithm to obtain corresponding symmetric solutions, skew-symmetric solutions, centro-symmetric solutions and so on. Based on the classical method, we will utilize the operable iterative method to find the symmetric arrowhead matrix solution of the matrix equation (1).

II. THE CONJUGATE GRADIENT ALGORITHM

In this section, by means of the study of the classical CG algorithm for solving the linear matrix equation in [4], we propose the following algorithm to solve the matrix equation (1) for the symmetric arrowhead solution and give some main results in detail.

Firstly, we define the following linear operator:

$$□\ □\mathbf{R}_{n\,n□}\ □\ \mathbf{AR}_{n\,n□}$$

$$□ : □ \tag{2}$$

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$$\square\square X \square \square\square X \square$$

in which $\square\square X \square \square E X_{11} \square XE_{11} \square diag\, X \square \square \square 2E\, XE_{11}$ $_{11}$.

According to the properties of the inner product matrix, it is easy to verify that

$$\langle X\, Y, \square\, X, \square \square Y \square \langle \square \square \square X \square, Y \rangle$$

in which $X \square \mathbf{R}^{n\, n\square}, Y \square \mathbf{AR}^{n\, n\square}$. Here we discuss the iterative algorithm of the matrix equation (1) as:

Algorithm CG-W.

Step1. Initialization. For initial matrix $X_1 \square \mathbf{SAR}^{n\, n\square}$, compute

$$R_1 \square C \square AX B_1,$$

$$P_1 \square \frac{A\, R\, B_{T\,1\,T} \square \square A\, R\, B_{T\,1\,T} \square_T}{2},$$

$$Q_1 \square \square\square \square P_1 \square E\, P_{11\,1} \square PE_{1\,11} \square diag\, P \square \square_1 \square 2E\, PE_{11\,1\,11}.$$

Step2. Iteration. For $k \square 1,2,\square$, compute

$$X_{k\square 1} \square X_k \square \frac{\|R_k\|^2}{\|Q_k\|^2} Q_k,$$

Step3. Compute

$$R_{k\square 1} \square \square CAX_{k\square 1}B,$$

$$P_{k\square 1} \square \frac{A\, R_{Tk\square 1}B_T \square \square A\, R_T\ _{k\square 1}B_T \square_T}{2},$$

$$Q_{k\square 1} \square\square \square P_{k\square 1} \square\square \frac{trace\, P\, Q \square\ _{kT\square 1\, k} \square}{\|Q_k\|^2} Q_k.$$

if $R_{k\square 1} \square 0$ or $R_{k\square 1} \square 0$, $Q_{k\square 1} \square 0$, stop; otherwise continue to step (2).

By algorithm CG-W it is clear that:

$$P_i \square \mathbf{SR}_{n\, n\square},\ Q_i \square \mathbf{SAR}_{n\, n\square},\ X_i \square \mathbf{SAR}_{n\, n\square},\ i \square 1,2,\square.$$

The following will demonstrate that the algorithm CG-W is terminated by a finite iterative step.

Lemma 1. The sequences $\square R_i \square$ and $\square Q_i \square$ generalized by Algorithm CG-W satisfy

$$\square\ ^T R_j \square \square\ trace\, R\, R \square \frac{\|R_j\|^2}{\|\square\|} \square R_{i\,2}\, trace\, Q\, P \square\ _{ij} \square,\ ,i$$

$j \square 1,2,\square$ (3)

$trace\ R_{i\square 1}$

Q_i

Proof: Since $P_i^T \square P_i$, $Q_i^T \square Q_i$, by Algorithm CG-W, we have

$$trace\, R\, R \square\ _{iT\square 1\, j} \square \square\ trace \square\square C \square AX_{i\square 1}B \square_T \square \square C \square A\, X \square\ _i \square R_{i\,22}\, Q_i \square\square B \square\square R_j \square\square\square$$

$$R_j \square \square\ trace \square\square\square\square\square \quad \square\square \quad Q_i \quad \square\square \quad \square\square \quad \square\square$$

$$: \quad \frac{\|R\|^2}{\|Q_i\|^2} BQAR^T$$

$$\frac{\|R\|^2}{\|Q_i\|^2} trace \frac{QARB_j\ ^T \ .QAR\ _j B^T\ ^T}{2}$$

$$\frac{\|R\|^2}{\|Q_i\|^2} trace \frac{Q_i : ARB_j\ ^T \ .ARB_j\ ^{T\ T}}{2}$$

$$\frac{\|R\|^2}{\|Q_i\|^2} trace QP_{i\ j}.$$

$$\square$$

$$\square\ trace \square\square C \square AX\, B \square\ R \square$$

$$\square\ trace\, R\, R \square\ _{iT} \quad _j \square\square \quad \square\square$$

$$\square\ trace\, R\, R \square\ _{iT} \quad _j \square\square \quad \square\square\square$$

□ trace R R□ $_{i}^{T}$ $_{j}$□□

Lemma 2. For k □ 2, the sequences □R_i□ , □Q_i□generalized by Algorithm CG-W satisfy

trace R R□ $_{iTj}$ □ □ 0, trace Q Q□ $_{iTj}$ □ □ 0, $i\ j$, □1,2,□, ,k i □ j (4)

Proof: We shall prove this lemma by induction.

First, notice that P_i □**SR**$^{n\ n}$□, Q_i □**SAR**$^{n\ n}$□, by Lemma 1 and Algorithm CG-W, we obtain

$_{2\ 1}^{T}$ trace R R□ $_{1\ 1}^{T}$ □□ $R^{1}{}_{22}$

trace Q P□ $_{11}$□

trace R R□ □□ ‖ ‖ ‖Q_1‖R_{12} □ R_{122} $_{111}^{T}$

□ trace‖QQ‖□trace $\|R_1\|^2 \cdot \frac{\|R\|^2}{\|Q_1\|^2}$ Q Q□ □□ 0,

Q_1

as well as trace Q Q□ $_{2}$ $_{1}^{T}$ □ □ ‖Q_1‖

trace□□□□□□□□□□□P$_2$□□ trace P Q□ $_{22T1}$□Q_1□□□□$_T$$Q_1$□□□□□

□ trace □□□P$_2$□□$_T$ Q_1□ □trace PQ□ $_{21}$□ □ 0.

Suppose that (4) holds for k □ □s 2 and notice that trace Q Q□ $_{sT}$ $_{s□1}$□ □ 0 .According to Lemma 1, we have

trace R • traceRR$_s^T$ $_s$ • $\frac{\|R\|^2}{\|Q_s\|^2}$ R□ $_{s□1s}$ trace Q P□ $_{ss}$□ □ R_s□ $_{2}$trace ‖ $_s$‖^2 · $\frac{\|R\|^2}{\|Q_s\|^2}$traceQ Q□ $_s$□□P_s□□ ‖ ‖ ‖Q_s‖ ‖ $_s$‖^2 · $\frac{\|R\|^2}{\|Q_s\|^2}$ · □ R_s□□Q_s□ trace P Q□ $_{sT2}$ $_{s□1}$□$Q_{s□1}$□□□□ ‖ $_s$‖^2 · $\frac{\|R\|^2}{\|Q_s\|^2}$traceQQ ‖□□‖ $Q_{s□1}$ □□□□

$_{T}$ trace P Q□ $_{sT}$ $_{s□1}$□ $_{T}$ □ □ R □trace Q Q$_{ss}$ □□ $_{2}$ trace Q Q□ $_s$ $_{s□1}$□□ $Q_{s□1}$ □□

□ R $_{s}^{T}{}_{s}$□ □ 0,

and

trace Q□ $_{s}^{T}{}_{□1}Q_s$ □ □ trace□□□□□□$P_{s□1}$□□ trace P Q□ $_{sT□1t}$ □Q_s□□$_T$$Q_s$□□ $\|Q_s\|^2$ □□□□□ □□□□□

□ trace□□□□□$P_{s□1}$□□$_T$$Q_s$□□□trace P Q□ $_{s}^{T}{}_{□1s}$□ □ 0.

Thus, by Lemma 1, it is clear that trace R□ $_{s}^{T}{}_{□1}R_j$ □ □ 0 when j □1. And we notice that trace R R□ $_{s}^{T}{}_{j}$□ □ 0 , trace Q Q□ $_{s}^{T}{}_{j}$□ □ 0 , trace Q Q□ $_{s}^{T}{}_{j□1}$□ □ 0 for j □ 2,3,□,s□1 . By Lemma 1 and Algorithm CG-W, we have

$$traceR_{s+1}^{T}R_j = traceRR_s^T{}_j - \frac{\|R\|^2}{\|Q_s\|^2}traceQP_s{}_j - \frac{\|R\|^2}{\|Q_s\|^2}traceQ_s P_j$$

$$- \frac{\|R\|^2}{\|Q_s\|^2}traceQ_s Q_j - \frac{tracePQ_j^T{}_{j-1}}{\|Q_{j-1}\|^2}Q_{j-1}$$

$$- \frac{\|R\|^2}{\|Q_s\|^2}traceQQ_j - \frac{tracePQ^T_{j-1}}{\|Q_{j-1}\|^2}traceQ_s^T Q_{j-1} = 0,$$

and trace Q□ $_{s}^{T}{}_{□1}Q_j$□ □ trace□□□□□□P □□ trace P Q□ $_{sT□1s}$□ $\frac{T}{\|Q_s\|^2}$ $_s$ □□ Q □ Q $_{s□1j}$ □□□ □□□□□ $_j$ $\frac{tracePQ_1^T{}_s}{\|Q_s\|^2}$

□ trace□□□□□$P_{s□1}$□□$_T$Q □trace Q Q□ $_{s}^{T}$ $_{j}$□

$$\square\ trace\ P\ Q \square_{s\square 1 j} \square\ \square\ trace\ Q\ P \square_{j\ s\square 1} \square .$$

which gives

$$trace\ Q \square_{sT\square 1} Q_j \square\ \square\ \frac{\|R_i\|^2}{\|Q_j\|^2}\ trace\ Q\ P \square_{j\ s\square 1} \square$$

$$\square\square\square trace\ R\ R \square_{Tj}\ {}_{s\square 1} \square \square trace\ R \square_{Tj\square 1} R_{s\square 1} \square \square\square$$

$$\square \frac{\|R_i\|^2}{\|Q_j\|^2} \square\square trace\ R \square_{sT\square 1} R_j \square \square trace\ R \square_{sT\square 1} R_{j\square 1} \square \square\square\ \square\ 0,$$

Thus, (4) holds when $k \square \square s$ 1. By the induction, we know that (4) holds for when $,i\ j \square 1,2,\square,\ ,k\ i \square j$.

Lemma 3. Suppose that the equation is consistent and X^* is one solution of (1), then the sequences $\square R_i \square$ and $\square Q_i \square$ generalized by Algorithm CG-W satisfy

$$trace\square\square \square X^* \square X_k \square Q_k \square\square\ \square\ \|R_k\|^2,\ k \square 1,2,\square \qquad (5)$$

Proof: We shall also prove this lemma by induction. First of all, when $k \square 1$ we have

$$trace\square \square X^* \square X_1 \square Q_1 \square\square\ \square\ trace\square\square \square X^* \square X_1 \square \square\square\ \square P_1 \square\square\ \square\ trace\square\square \square X^* \square X_1 \square P_1 \square\square$$

$$\square\ trace\square\square \square X^* \square X_1 \square \square\square\ A\ R\ B_{T\ 1\ T}\ \square 2 \square A\ R\ B_{T\ 1\ T} \square_T \square\square\square\square\square\square\ \square$$

$$\square\ trace\square \square \square BR\ A\ X_{1T} \square {}^* \square X_1 \square \square\ \square \square \square X^* \square X_1 \square BR\ A_{1T} \square \square\square$$

$$\square\ trace\square \square \square A\ X \square {}^* \square X_1 \square BR_{1T} \square_T \square \square A\ X \square {}^* \square X_1 \square BR_{1T} \square \square\square$$

$$\square\ trace\ A\ X \square\square \square {}^* \square X_1 \square BR_1{}^T \square\square\ \square\ trace\square\square \square C \square \|AX B\|\ R_1 \square {}_1{}^T \square\square\ \square\ trace\ R\ R \square_{1\ 1}{}^T \square\ \square\ \|R_1\|^2,$$

Suppose that (5) holds when $k \square s$. Owing to

$$trace\square\square \square X^* \square X_{s\square 1} \square Q_s \square\square\ \ \frac{\|R_i\|^2}{\|Q_s\|^2} Q_s : Q_s :\ \square$$

$$trace\square\square\square\square\square\square\ X^* \square X_s \square\ \ \cdot\cdot \|R_s\|^2 \cdot \frac{\|R\|^2}{\|Q_s\|^2}$$

$$\square\ trace\square\square \square X \square X_s \square Q_s \square\square\square\ Q_{s\,2}\ trace\ Q\ Q \square_s\ {}_s trace\ Q\ Q \square_s\ {}_s \square\ \square\ 0,$$

we obtain

$$\square\ trace\square\square \square X \square X_{s\square 1} \square Q_{s\square 1} \square\square\ \square\ trace\square\square \square X \square X_{s\square 1} \square \square\square\square\square P_{s\square 1} \square\square Q_s \square\square\square\square\ \frac{trace\ P\ Q \square_{sT\square 1\ s} \square}{\|Q_s\|^2}$$

$$\square\ trace\square\square \square X \square X_{s\square 1} \square \square\square P_{s\square 1} \square\square\square\square\ \frac{trace\ P\ Q \square_{sT\square 1\ s} \square}{\|Q_s\|^2}\ trace\square\square \square X \square X_{s\square 1} \square Q_s \square\square$$

$$\square\ trace\square\square \square X^* \square X_{s\square 1} \square_{s\square 1} \square\ \square\ trace\square \square X^* \square X_{s\square 1} \square \square\square\ \square A\ R_T\ B_T\ {}_{s\square 1} \square 2 \square A\ R_T\ {}_{s\square 1} B_T \square_T \square\square\square\square\square\square\ P \square$$

$$\square\ trace\square \square \square BR_{sT\square 1} A\ X \square {}^* \square X_{s\square 1} \square \square_T \square \square \square X^* \square X_{s\square 1} \square BR_{sT\square 1} A \square \square\square$$

$$\square\ trace\square\square \square C \square AX\ \ {}_{s\cdot 1} BR^T_{s\cdot 1} : \cdot\ trace R_{s\cdot 1} R^T_{s\cdot 1} \cdot\cdot \|R_{s\cdot 1}\|^2 .$$

By the induction, we know that (5) holds for $k = 1, 2, \ldots$.

Theorem 1. Suppose that the matrix equation (1) is consistent and for any initial matrix $X_1 \in \mathbf{SAR}^{n \times n}$, the sequence $\{X_k\}$ generated by Algorithm CG-W converges to a solution of (1) after finite-steps.

Proof: The proof is by contradiction. Assume that $R_i \neq 0$, $i = 1, 2, \ldots, mp$, then by Lemma 3, we have $Q_i \neq 0$ $i = 1, 2, \ldots, mp$ and can further obtain X_{mp+1} and R_{mp+1}. If $R_{mp+1} \neq 0$, then according to Lemma 2 we get the orthogonal basis matrix set $\{R_1, R_2, \ldots, R_{mp}, R_{mp+1}\}$ of $R^{m \times p}$, which contradicts the assumption. Thus, $R_{mp+1} = 0$, and X_{mp+1} is the exact solution of (1).

Theorem 2. Suppose that the matrix equation (1) is consistent, then we take the initial matrix

$$X_1 = \Phi(P_1), \quad P_1 = (A^T H B^T + B H^T A)/2,$$

with any $H \in \mathbf{R}^{m \times s}$ (or specially, for $X_1 = 0 \in \mathbf{R}^{n \times n}$), the Algorithm CG-W converges to the minimum norm solution of (1) after finite-steps.

Proof: If we take $X_1 = \Phi(P_1)$, $P_1 = (A^T H B^T + B H^T A)/2$ with any $H \in \mathbf{R}^{m \times s}$, by Algorithm CG-W, we can get a solution $\hat{X}$ of the matrix $AXB = C$ after finite-steps, and there exists the matrix $\hat{H} \in \mathbf{R}^{m \times s}$, such that $\hat{X} = \Phi(\hat{P})$ with $\hat{P} = (A^T \hat{H} B^T + B \hat{H}^T A)/2$. From $A\hat{X}B = C$ we know that all the symmetric arrowhead solution of matrix equation $AXB = C$ can be expressed as $\hat{X} + \tilde{X}$ with $\tilde{X} \in \mathbf{SAR}^{n \times n}$, satisfying $A\tilde{X}B = 0$. For $\tilde{X} = \Phi(\tilde{X})$, we get $A\tilde{X}B = 0$, thus we have

$$\langle \hat{X}, \tilde{X} \rangle = \langle \Phi(\hat{P}), \tilde{X} \rangle = \left\langle \frac{A^T \hat{H} B^T + B \hat{H}^T A}{2}, \tilde{X} \right\rangle = \langle \hat{H}, A\tilde{X}B \rangle = 0,$$

thus we get

$$\|\hat{X} + \tilde{X}\|^2 = \|\hat{X}\|^2 + \|\tilde{X}\|^2 \geq \|\hat{X}\|^2$$

$\hat{X}$ is the symmetric arrowhead minimum norm solution of (1). It is not difficult to verify the solution set of (1) is a closed convex set, therefore, the symmetric arrowhead minimum norm solution of (1) is unique.

III. Numerical Experiments

In this section, under the compatibility condition of the constrained matrix equation $AXB = C$, we give an example to illustrate the efficiency and investigate the performance of Algorithm CG-W which has been shown to be numerically reliable in various circumstances. All functions are defined by Matlab 7.0 and all codes are calculated with machine precision around 10^{-9}.

Example 1. Given $A = [toeplitz(1:30*i)\ zeros(30*i, 11*i)]$ of row full rank, $B = [eye(40*i); ones(i, 40*i)]$ of column full rank for $i = 1, 2, \ldots 5$ and $C = A\bar{X}B$. Given $Y = 0.5ones(n, n)$ and $\bar{X} = E\bar{Y}_{11} + P\bar{Y}_1 + diag(\bar{Y}) - 2E\bar{Y}E_{11}$. Notice that in this case, the matrix equation $C = AXB$ is consistent and has a unique minimum norm solution.

TABLE I
The Iterative Steps, Iterative Time and Residual Norm of the Algorithm CG-W

		CG-W
$i = 1$	Iter	94
	CPU	0.282
	$\|R_k\|$	3.8626e-008
$i = 2$	Iter	249
	CPU	1.690
	$\|R_k\|$	4.1348e-008
$i = 3$	Iter	420
	CPU	7.995
	$\|R_k\|$	8.8017e-008
$i = 4$	Iter	609
	CPU	23.669
	$\|R_k\|$	9.5693e-008
$i = 5$	Iter	820
	CPU	62.574
	$\|R_k\|$	9.7150e-008

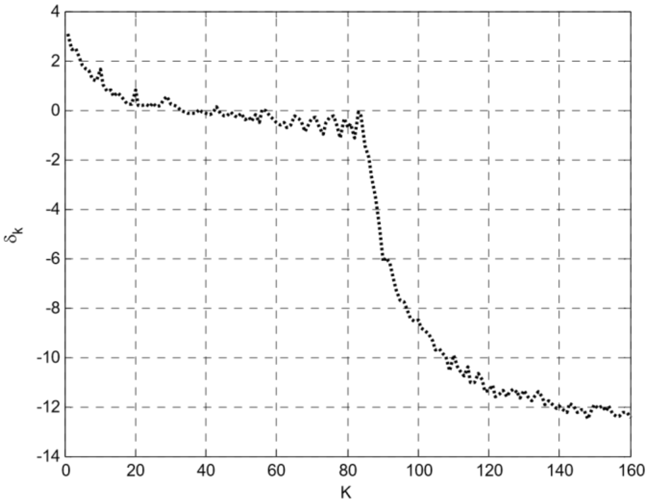

Fig. 1 Relation between error $\square_k$ and iterative number K when $i \square 1$

In Table I, we obtain iterative steps(Iter), Iterative time (CPU)

and residual norm $R_k \square\ AX\ B\ C_k \square \|_F$ of the algorithm respectively. We set a stop criterion for $R_k \square\ AX\ B_k \square^{C}{}_F$ $\square 10^{\square 7}$. Then, Fig. 1 plots the relation between error $\square_k \square$ log $(_{10} AXB \square C\|)$ and the iterative number K when $i \square 1$.

We choose the initial matrix $X_0 \square\ zeros(41^*, 41^*)i\ i$, the unique minimal norm solution of the matrix equation (1) is obtained by the algorithm CG-W. It can be seen from the Table I, when the order of the matrices A and B is growing exponentially, the iterative steps of the algorithm CG-W is growth multiples.

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