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Dynamics of Stratified Fluids: A Mathematical Exploration of Viscous Flow with Salinity and Heat Transfer

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Abstract—A model of the mathematical fluid dynamics which describes the motion of a three-dimensional viscous rotating fluid in a homogeneous gravitational field with the consideration of the salinity and heat transfer is considered in a vertical finite layer. The model is a generalization of the linearized Navier-Stokes system with the addition of the Coriolis parameter and the equations for changeable density, salinity, and heat transfer. An explicit solution is constructed and the proof of the existence and uniqueness theorems is given. The localization and the structure of the spectrum of inner waves is also investigated. The results may be used, in particular, for constructing stable numerical algorithms for solutions of the considered models of fluid dynamics of the Atmosphere and the Ocean.

Stokes equations, stratified fluid.

I. INTRODUCTION

LET us consider a bounded domain □ □ R^3 and the following system of fluid dynamics

□□□□ut_1□□ □u_2□ □ □u_1□□xp_1□ 0
□
□□u_2 □p
□ □t□□ □u_1□ □ □u_2□x_2□ 0
□
□□u^3 □p
□□ □ □□ u_3 □□□ □$_1$□ $_2W$□ 0
□ □t □x

□□div□ □ 0
$_3u$

□□□
□□□$_{33}u$□ 0
□□t
□□W
□□ □t□ □ □□ W□$_{43}u$□ 0
x□□, t□ 0.

Here u□ □ □u u u_1, $_2$, $_3$□ is a velocity field, $p(x\ t,)$ is the scalar field of the dynamic pressure, □□x,t□ is the dynamic density of the fluid, $W\ x\ t$□ , □ is either dynamic salinity or dynamic temperature, □□ Const is the Coriolis parameter, and

□$_i$, i □ 1,...4 are constant nonzero stratification parameters.

For the kinematic viscosity coefficient □ ,we assume □□ 0 .

The considered equations are deduced, for example, in [1].

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The study of mathematical properties of different systems of fluid dynamics of rotating fluid was started in [2]-[4]. Various problems involving the spectrum of normal vibrations for stratified and rotating fluid were considered in [5]-[10]. For non-linear model considered in bounded domains, but without the equations for salinity and heat transfer, the solution of similar systems was studied in [11]. We can observe that, for some problems of Ocean and Atmosphere dynamics, particularly for the cases when the horizontal dimensions are considerably larger than vertical dimensions, the third equation of the previous system does not contain the terms □u^3 and □u_3 (see, for example, [12]). Therefore, we will □t consider the system

□□v^1□□v_2□ □ □□ v_1□p□ 0
□ □t □x
□$_1$□□v_2 □p
□ □t□□v_1□ □ □□ v_2□x_2□ 0
□
□□p
□□□$_{14}v$□□$_{25}v$□ 0 (1)

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□□x_3

□□div$v$□ □ 0

□□v_4
□□□$_{33}v$ □ 0
□ □t
□□v_5
□□ □t □□□ □v_5□$_{43}v$ □ 0 x □□, t □ 0

in the domain

Q □ □□□t □ 0□, □ □ □x □ □x x□, $_3$□ □ □x x
x_1, $_2$, $_3$□, x□□R^2, 0 □ x_3 □ h□.

We will consider the initial conditions
$v_i|_{t□0}$ □ v_i^0 □x□ , i □1,2,4,5   (2)

and boundary value conditions

□v_i / □x_3 $|_{x_3□0}$ □ 0 , i □1,2 ; $v_i|_{x_3□0,\ x_3□h}$ □ 0 , i □ 3,4,5 .   (3)

II. Problem Formulation

Our primary aim is to construct the solution of the problem (1)-(3). The general idea of construction of such solution in a layer is taken from [14].

We will use the Laplace transform with respect to t, the Fourier transform with respect to x□ and finite integral transforms with respect to x_3. We apply the Cosine-Fourier transform to the first, the second and the fourth equations of (1), and the Sine-Fourier transform to the rest of the equations. For that purpose, we multiply the first, the second and the fourth equations by cos□$_n x_3$, the rest of the equations we multiply by sin□$_n$ x_3 , and integrate with respect to x_3 on the interval 0 □ x_3 □ h . Let us introduce the following notations:

□$_n$ □□n/h,

□v p ^$_i$, ^□□x n t□, , □ □ □□v p_i, □□x x t□, $_3$, □cos□$_n x$ $dx_{3\,3}$, i □ 1,2, (integral from 0 to h)

□v v v ^$_3$, ^$_4$, ^$_5$ □□x n t□, , □ □ □□v v $v_{3,\,4,\,5}$ □ □x x t□, $_3$, □sin□$_n x$ dx_3 $_3$, (integral from 0 to h)

□v v v ^$_i$, ^$_4$, ^$_5$ □□x n t□, , □$|_{t□0}$ □ □v v ^$_i^0$, ^$^0_4$,v ^$_5^0$ □ □x n□, , , □ i □1,2.

Using the boundary value conditions (3), we transform the problem (1)-(3) into the following:

□□v^$_1$□□ □ □□v^$_2$□ □$_2v$^$_1$□ $_{n2}v$^$_1$□ □p^ □ 0
□ □t □x
□$_1$
□□v^$_2$ $_2$ □p^
□ □t □□ □v^$_1$□ □$_2v$^$_2$□□□$_n v$^$_2$□ □x_2□ 0
□
□□□ □ □$_n p$^ □ $_1v$^$_4$□ $_2v$^$_5$
□
□□v^$_1$□v^$_2$ (4)
□ □ □□$_n v$^$_3$□ 0
□□x_1 □x_2
□□v^$_4$
□ □□$_3v$^$_3$□ 0
□ □t
□□v^$_5$ $_2$
□ □t □ □□ $_2v$^$_5$□□□ □$_n v$^$_5$□ $_4v$^$_3$□ 0,
□

□v v v^$_i$, ^$_4$, ^$_5$ □□x n t□, , □$|_{t□0}$ □ □v v^$_i^0$, ^$^0_4$,v^$_5^0$ □ □x n□, , , □ i □ 1,2. (5)

To solve the problem (4), (5), we assume that the initial conditions are sufficiently smooth and rapidly decreasing functions for x□ □ □ , which allows us to apply the Fourier transform in x□ and Laplace transform in t .

After introducing the notations

$F_{x□□□}$ □□$L_{t□}$ □□$v\ p\ v\ v$□^, ^ ^, 4, ^5 □□□$x\ n\ t$□, , □ □
$L_{t□□}$ □□v□, p, v_4, v_5 □□□□□, ,$n\ t$□ □

□ □$v\ p\ v\ v$□□, □ □ □, 4, 5 □□□ □□, ,n □,

$F_{x□□□□}$ □□$v\ v\ v$^$_i$o, ^o4, ^5o □□□$x\ n$□, □ □ □$v\ v$□
□ □$_i$o, o4,v_5o □□□□,n□ , i □1,2,

we obtain the system of algebraic equations

□□□□ □□ □ □□□2 □ n^2 □v□1 □
v□2 □ i $_1p$□ □ v□1o

□□□□ □□□□2 □n^2 □v□2 □□ □v□1 □ i
$_2p$□ □ v□2o
□ □ □$_n p$□ □ $_{14}v$□ □ $_{25}v$□ □ o (6) i
v□ □ □$_{11}$□ □ i $_{22}v$□ □ $_n v$□3 □ o
□ □v□4 □ $_{33}v$□ □ v□40

□□□□ □□ □□□2 □ n^2 □v□5 □ 4
$_3v$□ □ v□5o.

Let us introduce the functions

R_i
□$_i$ □□ □□, ,n □ □
—, i □ 0,1,2, (7)
□

where
R □ □ □□ □□□□□2 □ n^2,

□ □ R □□ □□ □□$_{n2\,2}R$ □ $_{2\,2n}$
□□2 □,

□ □□ □□□ 13 □ 24.

From (7), we can represent the inverse Laplace transform

for the functions □$_i$ as follows.

□0 •• •, nt, • • $\dfrac{2e^{\dots}}{n}$ sin^2 $\dfrac{t}{2n}$,

□1 •• •, nt, • • $\dfrac{2e^{\dots}}{n}$ sin $\dfrac{t}{n}$,

□2 •• •, nt, • • $\dfrac{2e^{\dots}}{n^2}$ cos $\dfrac{t}{n}$,

•• $\sqrt{\dots^2\ n^2\ \dots}$.

For the following, we assume v_io □W_1^4 □□□ , i □1,2,4,5,

h o o
□□v_1 □v_2 □
□o □ □x_1 □ □x_2 □□dx_3 □ o ,
□

We also suppose that the condition of consistency of the initial data and boundary values is fulfilled.

After solving (6) and applying the inverse Fourier and Laplace transforms F□□□□1 $_x$□L□□1□$_t$, we can represent the solution of the problem (4)-(5) as

v^$_k$ □$x\ n\ t$□, , □ □___1 2 □ □$e^{i\,x}$□ □,□□□ □$v\ e$□$_{ko}$
$_{Ht}$ □ □□□ □□$_{k2}$ □ $_{2n2}$ □v□$_{ko}$□ □o
□2□□ $_{R^2}$

□ □□ 1□$_k$ □□□□$_{n2}$□ □ □1 □ 1□$_k$ □□□1 2 □ o
□□ v□$_{30□k}$ □

□□ □$_n$ □□i_k □ □ □1 □ 1□$_k i$□ □$_{3□k}$ □ o
□□U□3o □d□□ k □ 1,2,
□ □ 1$_{i\,x}$□ □,□□□ □□□ □$_n$ □ U□2o□ □1
U□1o□2 □□□□d□□,

$v^{3}x\ n\ t$□, ,□ □2□□2 □□□$_{k_2}e$ □□□ □□ 2U□30□1
□□

$p\ x\ n\ t$^ □ □, , □ □___1 2 □ □$e^{i\,x}$□ □,□□□
□□□□□ U□20□ □oU□10□1 □□

□2□□ $_{R_2}$

□□ □$_n$ □ ^{2}□ □ □$_0$

$_2$ □U□$_3$^{0}

□□d□□,

$\hat{v}_4$ □$x\ n\ t$□, , □ □ ___1 $_2$ □□| e_{ix}□ □,□□□ □□$v\ e$□$_{40\ Ht}$

□ □□ $_2$ □$_0$ □□$_4U$□ $_{40}$ □□v□$_{40}$ □ □

□2□□ $_{R_2}$

□□□$_{3n}$ □U□$_1$^{0}□ □$_1$ □U□ $_2$^{0}□$_0$ □ □d□□,

$\hat{v}_5$ •$x\ n\ t$•, , • • ___•2•$_2$ •• $_{R_2}$ e $v\ e_5$ | | •0 •$_3U_4$ v_5 • •

•••$_4$ $_n$ •U_1^{0}• •$_1$ •U $_2$^{0}•$_0$ • •d••,

where

1 $_{ix}$□ □,□□□ □ $_0$ $_{Ht}$ $_2$ □ $_0$
$_0$

U□$_1$^{0} □□□,n□ □ $i\ v$□$_1$ □$_1$^{0} □ i□$_2$^{v}□$_2$^{0} , U□ $_2$^{0}

□□□,n□ □ i v□$_1$ □$_2$^{0} □ i□$_2$^{v}□$_1$^{0} ,. U□ $_3$^{0}

□□□,n□ □□$_3$^{v}□$_4$^{0} □□$_4$^{v}□$_5$^{0} , U□ $_4$^{0}

□□□,n□ □□$_4$^{v}□$_4$^{0}□□$_3$^{v}□$_5$^{0},

H □ □□ □ □ □ □ ^{2}□ $_{n^2}$ □.

In this way, the solution of the problem (1)-(3) can be represented as follows ([13]):

1 2 □

□$v\ p_i$, □□$x\ t$, □ □ _h□$v\ \hat{p}_i$, ^□□x□,0,t□□ _

h□$_{n□1}$□$v\ \hat{p}_i$, ^□□$x\ n\ t$□, , □cos□□$_n x_3$□,

i □1,2,

2 □

□$v\ v\ v_{3\,4\,5}$,, □□$x\ t$, □ □ _h□$_{n□1}$□$v\ v\ \hat{v}_{3\,4\,5}$, ^ , ^

□□$x\ n\ t$□, , □sin□□$_n x_3$□.

(8)

We denote Q□ □ □□□o □ t □□□,

□

U $_o$ □$x\ x$□, $_3$ □ □ □$v\ v\ v\ v_{10}$, $_{20}$ $\|f\|_k \cdot \|f\|$, $_{40}$, $_{50}$

□□$x\ x$□, $_3$ □ , $_{W_k}$□ □□,

V $\hat{Q}$ □ $_□$ □□ {v_i □^{C}□□o,□□,L_2 □□ □□□

L_2 □□o,□□,W_2^{1}□□□□, i □1,2,

v_3 □L_2 □□ $_{o_2}$^{1} □ □□ o ,

□o,□□,W □ □□ □ , div$v$

□ □

□ $_{o1}$ □ v_i

□^{C}□□o,□□,L_2□ □□ □□ L_2

□□o,□□,W_2 □ □□ □, i □ 4,5 },

□ □

$V\ Q$□ $_□$□ □ {□$v\ v$$^□$, $_4$,v_5□□$V\ \hat{Q}$ □ $_□$□: $D\ v_t$

$_i$□ L_2 □$Q$$_□$□ , i □1,2,4,5},

□$_1$ □$_2$

□ □$_1$, □ □$_2$.

□$_3$ □$_4$

We define a *strong solution* of the problem (1)-(3) as a system of the functions □$v\ p\ v\ v$$^□$, , $_4$, $_5$□ such that

v_i □$C_{x\,t2,,1}$ □Q□□$C_{1x\,t,,o}$ □Q□ , i □1,2, p□$C_{1x\,t,,o}$

□Q□,

v_3 □$C^1_{x\,t',}$^{o}□ □Q □$C\ Q$□ □ , v_i □$C_{x\,t^2,}$^{1}□ □Q □C

Q□ □ , i □ 4,5

satisfy (1) and the conditions (2), (3).

We define a *weak solution* of the problem (1)-(3) as a system of the functions □$v\ v\ v$□, $_4$, $_5$□□$V\ Q$□ $_□$□ which satisfy the condition (2) and the integral identity

$_2$ □v_i□v_4 □$v_{5\,2\,3}$ □v_i □□$_i$ □{□ __□ □ □$_{i\,1}$

__□ □ □$_{4\,2}$__□ □$_5$ □□□ ____ □

$_{Q□\,i□1}$ □t □t □t $_{i□1\,j□1}$ □x_j □x_j

$_2$ □v_4 □□$_4$ □v_5 □□$_5$ □v_4 □□$_4$ □v_5 □□$_5$

□□ □(□$_1$□ □$_2$)□□(□$_1$□ □$_2$)□ $_{i□1}$ □x_i □x_i □x_i □x_i

□x_3 □x_3 □x_3 □x_3

$\square\square\square v_1\square\ \square_2 v_2\square_1\square\square\square_1\square v_3\square\ \square_4 \quad v_4\square_3\square\square\square_2$
$\square v_3\square\ \square_5 v_5\square_3\ \square\}dxdt\ \square\ 0$

for all $t\ \square\square 0,\square\square$ and for every vector function

$$\square\square x,t\square\ \square\ \square\square\ _i\square_{5i\square 1}\square V\,\hat{Q}\ \square\ _\square\square.$$

Our aim now is to study the properties of existence and uniqueness of the strong and weak solutions for (1)-(3).

III. Problem Solution

Theorem 1 The system of functions (8) defines a strong solution of the problem (1)-(3).

Proof. Evidently, it is sufficient to show that the series (8) converge uniformly with respect to x and t, together with their term-by-term derivatives in x and t , and that the initial conditions (2) are satisfied. Let us investigate the first component of the solution, since the rest of the components are analogous. For $\square\ \square\ 2$, $t\ \square\ t_0\ \square\ 0$, the derivatives of the series which define $v_1\square x\ t,\ \square$, are estimated in the following way:

$$D\,v\,x\,n\,t\square\hat{}_1\square\ \square,,\ \square\cos\square_n x_3\square\ C\,n_0\square_3\square\ e_{\square\square\ \square\ \square t}\square$$
$$\square_{2\square\ n2}\square\ \square\square\square\square\{v\square_{10}\square\ v\square_{20}\square \quad (9)$$
$$\square\ v\square_{40}\square\ v\square_{50}\}d\square\square\ \square\ Cn\|t\square\|\ \square\square\square\square_1\square \ \ U\square_0\square\ C\,n\,e_1\square\ \square\square_3\square\ e^{n^2 t_0}\ 2$$

Similarly,

$$Dv\,x\,n\,t_t\hat{}_1\square\ \square,,\ \square\ \square\ C\,e\square\ \square\square\ \square\square_t\square\ \square_{2\square\ n2}\square$$
$$\square\square\square\ \square\square_2\square\ _{n2}\square\square\square\{v\square_{10}\square\ v\square_{20}\square \qquad \square_n\square(10)$$
$$\square\ v\square_{40}\square\ v\square_{50}\}d\square\square\ \square\ C_1\square_1\square\ n_2\square e_{\square\square\square n^2 t_0}U\square_0.$$

We observe that the constants C_1 in (9) and (10), in general, depend on t_0. Due to the arbitrary choice of t_0 $\square\ 0$, it follows from (9), (10), that the series (8) converge uniformly in x and t, together with the series obtained as a result of term-by-term differentiation with respect to x and t.

Let us prove that $v_1\square x\ t,\ \square$ satisfies the initial condition (2).

For that, we represent the general term of the series as follows.

$$\frac{2\ \square\square_{n,0}}{h}\hat{v}_1\square x\,n\,t\square,,\ \square\ \square\ \square_{k\square 1}\hat{v}_{1,k}\square x\,n\,t\square,,\ \square,$$
$$2\ \square\square_{n,0}\ 1$$
$$_{i\,x}\square\ \square,\square\square\square e\square\square\ \square\ \square_t\square\ \square_{2\square\ n2}\square v\square_{10}\ \square\square\square,n\,d\square\ \square\square\ \square$$
$$\hat{v}_{1,1}\square x\,n\,t\square,,\ \square\ \square\ \frac{2\ \square\ e}{h}$$
$$\square 2\square\square\ _{R_2}$$

$$\square\ \_\_2\ \square\square_{n,0}\ \square\square\square\square\square_h\ v_{10}\ \square x\ x\square,\ _3$$
$$\square\cos\square_n x\,dx_3\ \ _3\ \square\square\ _R\square_2\ G\ x\ t\square\ \square,$$
$$\square\square\square e_{\square\square\square n^2 t},h\ \ \square\square\square\ _0\square\ \ \square\square$$

where $\square_{ij,}$ is the Kronecker symbol and $G\,x\,t\square\ \square,\ \square$ is the singular solution of the heat transfer equation.

Since

$$\square\cos\square_n x_3\square\ \square\ h \quad {}^2$$
$$n\square 0$$

$$\square\ W\ \square\square\square,n\square\ \frac{C}{\square}\ W\ _3\ ,$$
$$n\square_1\square\ \square^\square\ \square$$

and $W\ x\square\ \square$ is any of the functions $v_i^0\ \square x\square$, i $\square 1,2,4,5$.

□ e□
R₂

□

□□ □
n□0

are satisfied, which completes the proof.

Theorem 2 The weak solution of the problem (1)-(3), is unique.

Proof. Let □v v v□, 4, 5□ be a weak solution of the problem (1)-(3) for
v_i^0 □x□ □ 0 , i □1,2,4,5.

Our aim is to verify that v_i □x t, □ □ 0 , i □1,2,3,4,5.

We take □v v v□, 4, 5□ as test functions □$_i$. In this way, we obtain

2

12 − □ i□2 v₂□ v₂□ □₂v₅₂ — □□□dx
□□□{□□□₂₃□□□□□xvᵢⱼ□□□□ □
□□□ i □₁₄
□□ 1 Q i□1 j□1 □
2 2

□□□₂(□₁□□□□vx₄ᵢ□□□ □ □₂□□□—
□□vx₅ᵢ□□□)□ (12)
i□1 □

□□(□₁□□□□□vx₄₃□□□₂
□ □₂
□□□□
□vx₅₃
□□₂
)}dxdt □
0. □

□v₄ □v₅ i □ 3, which

It follows from (12) that □ □ 0 , 1□
□x_i □x_i

implies v_4 □x t, □ □ v_5 □x t, □ □ 0 , due to the boundary conditions.

Additionally, it follows from (2) that

□v_i□ 0 , □□₂v dxᵢ²□ 0 for all t□□0,□□ ; i
□1,2,1□ □j 3 ,
□x j □ i□1

which implies v_i □x t, □ □ 0 , i □1,2. From the equation of

— □v₃
v x t₃□ , □ □□□x t□, □, continuity, we have that
□0. Therefore, □x₃
and from the boundary conditions, we finally obtain that
v x t₃□ , □ □0, and thus, the theorem is proved.

Theorem 3 The strong solution of the problem (1)-(3), is unique and belongs to the class V Q□ □□ .

Proof. Let us consider the component v x t₁□ , □ of the solution.

Using the Parseval formula and the explicit representation (8), we have

□

v x t₁□ , □ ₂L₂□ □□□ h1 □□□ v x^₁□ □,0,t□ ₂L₂□
□□‖□ 2□n□1 v x-n‖t^₁□ □‖, □‖₂L₂□ ‖□□□□□ □

□2□‖□²□ ‖ ₂ ‖ □₂ ‖ □
□ □ v₁□□□,0,t□ L₂□ □□□ 2□ v₁
□□□, ,n t□ L₂□ □□□.
h □ n□1 □

Let us estimate the general term of the last series. With the help of the obvious inequality □a b□ □₂ □2□a b₂□ ₂□ and the explicit form of the functions □$_i$, we obtain

$\|v_1$ □□□, ,n t□$\|_{L_2}$ □ C e□ □2□ □ □t□ □2□ n2□{v□10 2 □ . Thus, we

□ v□20 2 □ v□40 2 □ v□50 2}d□□.

R_2

From the last relation and the proof of Theorem 1, we have

$\|v_1$ □□□, ,n t□ $_{2L_2}$ □ U□ $_0$ $_3$ $Cn_2{}^1{}_R$ □ $_2$

□$_1$□d□□□$_2$ □$_2$ □ $Cn_2{}^2$, (13)

which implies that v_1 □x t, □□^{C}□□0,□□,L_2 □□□□.

Let □ □ R_2 □□0 □ t □□□. Analogously, for □ □1, we obtain

D v x t□x 1□ , □ $_{2L}$ Q_2□ □□ □h1□□□□1□□1,□3□ D v x□x□□^1□ □,0,t□ $_{2L_2}$□ □□ □2□$_n$□□1 □$_n$□3D v x nt□x□□^1□ □, , □ $_{2L_2}$□ □□ □□□□

$\|$ $_2$ $\|$ – $\|$ $\|$

□□2□h□ □□□1□□1,□3□ □ □□2$_2$ □□□ □3v_1□ □, ,nt□ $_{2_2}$ □□.

i□□□ v_1□□□,0,t□□2□□i□ □ □□□

n

□$_L$ □ □□ n□1 $_L$ □ □□ □

Due to the inclusion property $W_1{}^4$ □ □□ □ □$W_2{}^2$□ □, the general term of the series may be estimated as follows:

{ □

i□1,2,4,5

$_2$ □□d□□□

□

obtained that

u □ 0 x□□, t □ 0.

in a bounded domain □ □ R^3 with the boundary □□ of the classC^1. We associate system (13) to the boundary conditions

u n□ □□ | □ 0 (14)

□□

where n□ is the exterior normal to the surface □□. Let us consider the following problem of normal vibrations

u x t□□ , □ □ v x e□□ □ □□t

p x t□ , □ □ _^{1} v_4□ □x e□□t

□ (15)

□□x t, □ □ v_5□ □x e□□t

W x t□ , □ □ v_6□ □x e□□t , □□C.

We denote ^{v}□ □ □v v v v□, $_4$, $_5$, $_6$□ and write the system (13) in the matrix form

Lv□ □ 0 (16)

where

L □ M □□I

and

[illegible] $\; 1 \; 0 \; x_1 \; 0 \; 0 \; x_2 \; 0 \; 0 \; M \; 0 \; 0 \; \ldots_{12}. \; x_3 \; 1 \; 1 \; 1 \; 0 \; 0 \; 0 \; x_1 \; x_2 \; x_3 \; 0 \; 0 \; {}_3 \; 0 \; 0 \; 0 \; 0 \; 0 \; {}_4 \; 0 \; 0$ (17)

We define the domain of the differential operator M with the boundary condition (14) as follows.

$D(M)$ [illegible] $v \; W_{021} \; \ldots_3, v \; W_5 \; {}_{021} \; \ldots, v \; W_6 \; {}_{021} \; \ldots \; v_4 \; L_2 \; \ldots : Mv \; L_2 \ldots {}_6$.

The consideration of the separated variables of the form (15) permits to interpret every non-stationary process as a linear sum of the normal oscillations. The spectrum of inner vibrations may be used for investigating the properties of the stability of the solutions. As well, the spectral properties of M may be used in the studying of weakly non-linear flows, since the points of bifurcation are the points of the spectrum of the operator M.

We observe that the above defined operator M is a closed operator, and its domain is dense in $L_2 \ldots {}_6$.

Let us denote by $\sigma_{ess}(N)$ the essential spectrum of a closed linear operator N. We recall that, according to the definition of the essential spectrum [15], [16],

$\sigma_{ess}(N) = \{\ldots C : (N - \ldots I)$ is not of Fredholm type $\}$,

it consists of the eigenvalues of infinite multiplicity, limit points of the point spectrum, and the points of the continuous spectrum.

Therefore, the spectral points outside of the essential spectrum, are eigenvalues of finite multiplicity. For calculating the essential spectrum of M, we would like to refer to the property [17]:

$$\sigma_{ess}(M) = Q \cup S,$$

where

$Q = \{\ldots C : (M - \ldots I)$ is not elliptic in sense of Douglis-Nirenberg $\}$

and

$S = \{\ldots C \, Q \backslash$: the boundary conditions of $(M - \ldots I)$ do not satisfy Lopatinski conditions $\}$.

Theorem 4 The essential spectrum of the operator M is composed of one real point $\sigma_{ess}(M) = \ldots \frac{1}{2} \ldots$.

Proof. We observe that, according to the definition of the ellipticity in sense of Douglis-Nirenberg [18], the main symbol of the operator $L = M - \ldots I$ will be expressed as:

[illegible] $\; {}_2 \; 1 \; 0 \; 0 \; {}_1 \; 0 \; 0 \; 0 \; {}_2 \; 0 \; {}_2 \; 0 \; 0$

$L \ldots \; 0 \; 0 \; \ldots_2 \; {}_1 \; {}_3 \; 0 \; 0 \; \ldots$.

□ 1 1 1 □
· —·₁ · · —·₂ ·₃ ·· 0 0 ·
· 0 · · 0 ·· 0 ·
0 0
· ·
□□ 0 0 0 0 0 □□□□₂□□

We calculate the determinant of the last matrix:

det □M□□□I□ □ □□□□₃₂□ □□□₈□1□
₂□ ,
□

and thus, we can see that for only one point □□ 1/₂ □□ the operator L □ M □□I is not elliptic in sense of Douglis-Nirenberg. Now, we will show, additionally, that the conditions of Lopatinski [17] are satisfied.

The boundary condition (14) can be written in a matrix form

Gv□|□□ □ 0 , G □ □n_1 n_2 n_3 0 0 0□.

If we denote □□ □ □□□ □ □₁, ₂□ , ₃□ ; then

det □M□□□I□ □□□□, □ □ □₃₂ □ □ □□
₂□ ₂ □ ₄ □1□□□□₂ □ ,
□

and thus, the equation det □M□□□ □□I□ □ □, □ □ 0 for □□ 1/₂ □□ has one root □□ i □□ of multiplicity four in the upper half of the complex plane.

In this way, M□ □□□,□ □ □□ □ □ □i □□₄. Since the elements of the matrices M□□□I and G are homogeneous functions with respect to □□,□, then it is sufficient to verify the Lopatinski conditions for unitary vectors □□ . Let us choose a local system of coordinates so that □₁□1 , □₂□ 0 .

For the matrix □M□□□I□ , we construct first the adjoint matrix □M □□I□□ , then we multiply □M □□I□□ by the boundary conditions matrix G and thus obtain the following.

G M□□□I□□ □ □□n B₁³ □□B□□ □—₂₂
□□,0,□n B₃³—□₂,0,0,0 ,□□
□ □ □ □ □ □

where B □ □□□1□□₂ □ .

Since G M□ □□I□□ is a vector row, then evidently, the Lopatinski conditions are satisfied, and thus, the theorem is proved.

References

[1] G. Marchuk, *Mathematical Models in Environmental Problems*, North Holland, Elsevier, Amsterdam, 1986.

[2] S. Sobolev, On a new problem of mathematical physics, *Selected works of S.Sobolev*, Springer, N.Y., 2006.

[3] V. Maslennikova, The rate of decrease for large time of the solution of a Sobolev system with viscosity, *Math. USSR Sb.*, Vol. 21, 1973, pp. 584-606.

[4] V. Maslennikova, On the rate of decay of the vortex in a viscous fluid, *Proceedings of the Steklov Institute of Mathematics,* Vol. 126, 1974, pp. 47-75.

[5] V. Maslennikova, and A. Giniatoulline, Spectral properties of operators for the systems of hydrodynamics of a rotating liquid and nonuniqueness of the limit amplitude, *Siberian Math. J.*, Vol. 29, No.5, 1988, pp. 812-824.

[6] A. Giniatoulline, *An Introduction to Spectral Theory*, *Edwards, Philadelphia, 2005.*

[7] A. Giniatoulline, On the essential spectrum of operators, *2002 Proceedings WSEAS Intl. Conf. on System Science, Appl. Mathematics and Computer Science*, Rio de Janeiro: WSEAS Press, 2002, pp.12911295.

[8] A. Giniatoulline, On the essential spectrum of operators generated by PDE systems of stratified fluids, *Intern. J. Computer Research*, Vol. 12, *2003, pp. 63-72.*

[9] A. Giniatoulline, and C. Rincon, On the spect-rum of normal vibrations for stratified fluids, *Computational Fluid Dynamics J.*, Vol. 13, 2004, pp. 273-281.

[10] A. Giniatoulline, and C. Hernandez, Spectral properties of compressible stratified flows, *Revista Colombiana Mat.*, Vol. 41, (2), 2007, pp. 333-
344.

[11] A. Giniatoulline, On the strong solutions of the nonlinear viscous rotating stratified fluid, *Int. J. of Mat., Computational, Phys., Electr. and Computer Engineering,* Vol. 10, (10), 2016, pp. 469-475.

[12] G. Marchuk, V. Kochergin, and V. Sarkisyan, *Mathematical Models of Ocean Circulation*, Nauka, Novosibirsk, 1980.

[13] C.Tranter, *Integral Transforms in Mathematical Physics*, J. Wiley and
Sons, NY, 1971.

[14] V. Maslennikova, and I. Petunin, Asymptotics for $t\to\infty$ in the solution of An initial-boundary value problem in the theory of internal waves, *Diff. Uravneniya*, Vol. 31, No. 5, 1995, pp. 823-828.

[15] T. Kato, *Perturbation theory for linear operators*, Springer, Berlin,
1966.

[16] F. Riesz, and B. Sz.-Nag, *Functional Analysis*, Fr. Ungar, N.Y., 1972.

[17] G. Grubb, and G. Geymonat, The essential spectrum of elliptic systems of mixed order, *Math. Ann.*, Vol. 227, 1977, pp. 247-276.
[18] S. Agmon, A. Douglis, and L. Nirenberg, Esti-mates near the boundary for solutions of elliptic differential equations. *Comm. Pure and Appl. Mathematics,* Vol. 17, 1964, pp. 35-92.