

# Bi-Univalent Functions and Hohlov Operator: A Comprehensive Analysis

Amira Z. Saeed, Fatima M. Abdullah, and Nadia A. Hassan

Department of Mathematical Sciences, University of Qatar, Doha, Qatar; and Department of Pure and Applied Mathematics, University of Sharjah, Sharjah, United Arab Emirates

**Abstract**—The coefficients estimate problem for Taylor-Maclaurin series is still an open problem especially for a function in the subclass of bi-univalent functions. A function  $f \in A$  is said to be

bi-univalent in the open unit disk  $D$  if both  $f$  and  $f'$  are univalent in  $D$ . The symbol  $A$  denotes the class of all analytic functions  $f$  in  $D$  and it is normalized by the conditions  $f(0) = f'(0) - 1 = 0$ . The class of biunivalent is denoted by  $\Sigma$ . The subordination concept is used in determining second and third Taylor-Maclaurin coefficients. The upper bound for second and third coefficients is estimated for functions in the subclasses of bi-univalent functions which are subordinated to the function  $\phi$ . An analytic function  $f$  is subordinate to an analytic function  $g$  if there is an analytic function  $w$  defined on  $D$  with  $w(0) = 0$  and  $|w(z)| < 1$  satisfying  $f(z) = g[w(z)]$ . In this paper, two subclasses of bi-univalent functions associated with Hohlov operator are introduced. The bound for second and third coefficients of functions in these subclasses is determined using subordination. The findings would generalize the previous related works of several earlier authors.

## I. INTRODUCTION

**L**

ET denote the class of functions which are analytic in the open unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$  and normalized by the following Taylor-Maclaurin series:

$$\sum_{n=1}^{\infty} a_n z^n \in \Sigma \quad (1)$$

Recall that the convolution of two analytic functions  $f, g \in \Sigma$  is the analytic function defined as

$$f * g = \sum_{n=1}^{\infty} a_n b_n z^n,$$

where  $f(z) = \sum_{n=1}^{\infty} a_n z^n$  is given by (1) and  $g(z) = \sum_{n=1}^{\infty} b_n z^n$ .

For the complex parameters  $a, b, c$  and  $c \neq 0, -1, -2, -3, \dots$ , the Gaussian hypergeometric function is defined as

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n = 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (n-1)!} z^n \in \Sigma \quad (2)$$

where  $(\alpha)_n$  is the Pochhammer symbol defined, in terms of gamma function, by

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \quad (n = 0, 1, 2, 3, \dots)$$

Using the Gaussian hypergeometric function given by (2), Hohlov [1] introduced the operator  $\mathcal{H}_\alpha$ , as:

$$\mathcal{H}_\alpha f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n} z^n \quad \alpha \in \mathbb{C}$$

where

$$\mathcal{H}_\alpha f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n} z^n$$

In particular, if  $\alpha = 1$  then  $\mathcal{H}_\alpha$  reduces to the Carlson-Shaffer operator. Also, the Hohlov operator is a generalization of the Ruscheweyh operator and Bernadi-Libera-Livingston operator.

Let  $\Sigma_\alpha$  denote the subclass of functions in  $\Sigma$  which are univalent in  $U$ . According to the Koebe one-quarter theorem [2], it ensures that the images of  $\Sigma_\alpha$  under every univalent function in  $\Sigma$  contain a disk of radius  $\frac{1}{4}$ . Thus, every univalent function on  $U$  has an inverse, defined by

$$f^{-1}(z) \in \Sigma$$

and

$$f^{-1}(z) = \sum_{n=1}^{\infty} b_n z^n, \quad b_n = -\frac{a_n}{n}$$

A function  $f \in \Sigma$  is said to be bi-univalent in  $U$  if both  $f$  and  $f^{-1}$  are univalent on  $U$ . Let  $\Sigma^*$  denote the class of biunivalent functions in  $\Sigma$  given by the Taylor-Maclaurin



series expansion (1). Some examples of functions in the class are  $\frac{z}{1-z}$ ,  $-\log(1-z)$  and  $z$ .

In 1967, Lewin [3] developed the class of bi-univalent function and showed that  $|a_2| \leq 1.51$ . On the other hand, for the most general families of functions given by (1), the initial bounds for bi-starlike were conjectured in [4] that  $|a_2| \leq \sqrt{2}$  and  $|a_3| \leq 1$  for bi-convex functions [5]. The coefficient problem for each of the following TaylorMaclaurin coefficients  $|a_n|$  ( $n \in \mathbb{N} \setminus \{1,2\}; n = 1,2,3, \dots$ ) is still an open problem.

An analytic function  $f$  is subordinate to an analytic function  $g$ , denoted as  $f \prec g$  if there is an analytic function defined on  $\mathbb{D}$  with  $0 < \theta < 1$  and  $|\theta| < 1$  satisfying

$$f(z) = g(\theta z)$$

where all coefficients are real and  $0 < \theta < 1$ . The classes of MaMinda starlike and convex functions consist of function

$f \in \mathcal{S}^*$  satisfying the subordination  $\frac{zf'(z)}{f(z)} \prec \varphi(z)$  and  $1 + \frac{zf''(z)}{f'(z)} \prec \psi(z)$ , respectively.

Recently, the estimate for second and third coefficients of bi-univalent functions is investigated by [7]-[14]. Besides that, there are several authors who determined the initial bounds for the subclasses of bi-univalent functions associated with operator such as in [15]-[21].

Motivated by [12], [19], we introduce two new subclasses of bi-univalent functions associated with Hohlov operator based on Ma-Minda concept. Furthermore, the bound for second and third coefficients of functions in these subclasses are obtained. The results would generalize the previous related works of several earlier authors.

Using the Hohlov operator  $\mathcal{H}_\alpha$ , we introduce the following two subclasses of bi-univalent functions.

**Definition 1.** A function  $f \in \mathcal{S}$  is said to be in the class  $\mathcal{S}^*_\alpha$ , where  $\alpha$  is given in (3), if the following subordinations hold:

$$\frac{zf'(z)}{f(z)} \prec \varphi(z)$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \psi(z)$$

where

$$2 \leq \alpha < \infty \quad \dots \quad (4)$$

Note that for  $\alpha = 1$ , the class  $\mathcal{S}^*_\alpha$  reduces to the class introduced by Ali et al. [12]. On the other hand, if  $\alpha = \infty$  or  $\alpha = 2$  with  $\varphi(z) = z$  and  $\psi(z) = z$ , the subclasses introduced by [14] are obtained.

**Definition 2.** Let  $\mathcal{S}^*_\alpha$  is given in (3) and  $\alpha > 1$ . A function  $f \in \mathcal{S}^*_\alpha$  is said to be in the class  $\mathcal{S}^*_{\alpha, \beta}$ , if the following conditions are satisfied:

$$1 + \frac{zf'(z)}{f(z)} \prec \varphi(z); \quad \frac{zf''(z)}{f'(z)} \prec \psi(z)$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \psi(z); \quad \frac{zf'(z)}{f(z)} \prec \varphi(z)$$

where the function  $\varphi$  is given by (4).

For special cases, the class  $\mathcal{S}^*_{\alpha, \beta}$  reduces to the previous classes introduced by several authors. For examples:

- i) If  $\alpha = 1$  and  $\beta = 1$ , the class  $\mathcal{S}^*_{\alpha, \beta}$  reduces to [12].
- ii) If  $\alpha = 1$  and  $\beta = 1$  with  $\varphi(z) = z$  and  $\psi(z) = z$ , we obtain the subclasses defined by [14].
- iii) If  $\alpha = 1$ ,  $\beta = 1$  and  $\varphi(z) = z$ , the class  $\mathcal{S}^*_{\alpha, \beta}$  reduces to subclasses introduced in [13].

To establish the bounds for coefficients  $a_2$  and  $a_3$ , we state the well-known lemma that is used to obtain the bounds.

**Lemma 1.** If  $f \in \mathcal{S}$  then  $|a_n| \leq 2$  for each  $n$ , where  $\mathcal{S}$  is the family of all functions analytic in  $\mathbb{D}$ ,  $0 < \alpha < 1$  for  $f \in \mathcal{S}^*_\alpha$ .

## II. COEFFICIENT ESTIMATES FOR THE FUNCTION IN $\mathcal{S}^*_{\alpha, \beta}$

We begin by finding the bound for second and third coefficients for functions in the class  $\mathcal{S}^*_{\alpha, \beta}$ .

**Theorem 1.** Let  $f$  given by (1) be in the class  $\mathcal{S}^*_{\alpha, \beta}$  then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{1 - \alpha}}$$

and

$$| | \leq 1 \left[ \frac{B_1}{4\phi_2^2} + \frac{1}{3\phi_3} \right] \quad (5) \text{ and}$$

**Proof.** For  $\in \mathbb{C}$ ,  $z \in D$  and  $\phi_1, \phi_2, \phi_3$ , there exist analytic functions  $f, g, h : D \rightarrow \mathbb{C}$  with  $0 \leq \phi_1 < \phi_2 < \phi_3 < 1$ , satisfying

$$f(z) = \phi_1 + 2\phi_2 z + 3\phi_3 z^2 + \dots \quad (11)$$

Now, equating the coefficients from (8)-(11), we have

$$\text{and} \quad \phi_1 = \frac{2\phi_2}{3\phi_3} - \frac{B_1}{2} \quad (6) \quad (12)$$

Define the functions  $g$  and  $h$  as

$$g(z) = \frac{p_1^2}{2} \quad (13)$$

$$h(z) = \frac{1}{3\phi_3} \quad (14)$$

and

$$\text{and} \quad \phi_1 = \frac{3\phi_3}{2\phi_2} - \frac{1}{3\phi_3} \quad (15)$$

or, equivalently

From (12) and (14), we obtain

$$\text{and} \quad \phi_1 = \frac{32\phi_2^2 a_2^2}{6\phi_3} \quad (7) \quad (16)$$

Functions  $g$  and  $h$  are analytic in  $D$  with  $0 < \phi_1 < \phi_2$  and have positive real parts in  $D$ .

and

$$\phi_1 = \frac{32\phi_2^2 a_2^2}{6\phi_3} \quad (17)$$

It follows from (6) and (7), together with (3) that

Now, from (13), (15) and (17), we get

$$\text{and} \quad \phi_1 = \frac{6\phi_3}{4[3\phi_3 B_1^2 + 4\phi_2^2(B_1 - B_2)]} \quad (8)$$

where

$$\phi_1 = \frac{1}{4[3\phi_3 B_1^2 + 4\phi_2^2(B_1 - B_2)]} \quad (9)$$

and

$$\phi_1 = \frac{1}{4[3\phi_3 B_1^2 + 4\phi_2^2(B_1 - B_2)]} \quad (10)$$

On the other hand,

Next, by subtracting (15) from (13) and further computations, it leads to

$$6\phi - 6\phi - \dots$$

Then, it follows from (16) and (17) that

$$\frac{B_1 p_1}{16\phi_2^2} + \frac{B_1(p_2 - q_2)}{12\phi_3}$$

Applying Lemma 1 for coefficients  $\dots$  and  $\dots$ , yields

$$\left| \dots \right| \leq \left[ \frac{\dots}{4\phi_2^2} + \frac{\dots}{3\phi_3} \right]$$

which completes the proof of Theorem 1.

**Remark 1.** For  $\dots$  and 1, the results reduce to Theorem 1 in [12].

**Remark 2.** For  $\dots$  and 1, the class of strongly starlike functions, the function is given by

$$1 - 2\alpha z^2 + \dots, \quad (0 < \alpha \leq 1)$$

which gives  $\dots$  and  $\dots$ . Then, the inequalities (5) reduce to the result in [14, Theorem 1]. Furthermore, in the case

$$1 - 2\alpha z^2 + \dots$$

By letting  $\dots$ , the inequalities in (5) reduce to Theorem 2 in [14].

III. COEFFICIENT ESTIMATES FOR THE FUNCTION IN

In this section, we determine the bound  $\dots$  and  $\dots$  for functions in the class  $\dots$ .

**Theorem 2.** Let  $\dots$  be in the class  $\dots$ .

**1. Then**

$$\left| \dots \right| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{\dots}}$$

and

$$\left| \dots \right| \leq \frac{\dots}{(1+2\lambda)\phi_3} - \frac{\dots}{(1+\lambda)^2\phi_2^2}$$

**Proof.** For  $\dots$ , there are analytic functions  $\dots$  with  $\dots$ , such that

$$1 - \dots \tag{19}$$

and

$$1 - \dots \tag{20}$$

Since

$$1 - \dots$$

and

$$1 - \dots$$

Then, from (8), (9), (19) and (20), it follows that

$$1 - \phi \tag{21}$$

$$1 - 2\phi - \dots \tag{22}$$

$$1 - \phi \tag{23}$$

and

$$1 - 2\phi - \dots \tag{24}$$

From (21) and (23) yield

$$\tag{25}$$

Now, from (22), (24) and (25) lead to

$$4[B_1^2(1 + 2\lambda)\phi_3 + (B_1 - B_2)(1 + \lambda)^2\phi]$$

which yields the estimate on  $\dots$  as described in (18).

Proceeding similarly as in the earlier proof, making use (22)-(25) shows that

$$\overline{4(1+\lambda)^2\alpha_2^2} \quad \overline{4(1+2\lambda)\alpha_3} .$$

Then, applying Lemma 1 for coefficients  $\alpha_2$  and  $\alpha_3$ , we readily get

$$|\alpha_2| \leq \frac{\overline{4(1+2\lambda)\alpha_3}}{\overline{(1+\lambda)^2\alpha_2^2}}$$

which completes the proof of Theorem 2.

### Remark 3.

- i) For  $\lambda = 1$ , the result reduces to Theorem 1.
- ii) For  $\lambda = 1$  and  $\lambda = 1$ , the inequalities (18) reduces to Theorem 1 in [12].
- iii) If  $\lambda = 1$  and  $\lambda = 1$  with  $\alpha_2 = \alpha_3$  and  $\alpha_2 = \alpha_3$ , the inequalities (18) reduces to Theorem 1 and Theorem 2 in [14] respectively.
- iv) If  $\lambda = 1$ ,  $\alpha_2 = \alpha_3$  and  $\alpha_2 = \alpha_3$ , the result reduces to Theorem 2.1 and Theorem 2.2 given by [13].

### REFERENCES

- [1] Y. E. Hohlov, "Convolution operators that preserve univalent functions" *Ukrain. Mat. Zh.* vol. 37, pp 220-226, 1985.
- [2] P. L. Duren, "Univalent Function", *Springer, New York*, vol 259, 1983. [3] M. Lewin, "On a coefficient problem for bi-univalent functions" *Proc. Amer. Math. Soc.* vol. 18, pp. 63-68, 1967.
- [4] A. W. Kedzierawski, "Some remarks on bi-univalent functions" *Annales Universitatis Mariae Curie-Skolodowska Sectio A*, vol. 39, pp. 77-81, 1985.
- [5] D. A. Brannan and T. S. Taha, "On some classes of bi-univalent functions" *Studia Univ. babes-Bolyai Math*, vol. 31, no. 2, pp. 70-77, 1986.
- [6] W. C. Ma and D. Minda, "A unified treatment of some special classes of univalent functions" *Conf. Proc. Lecture Note Anal. I, Int. Press*, Cambridge, MA, 1994.
- [7] S. Bulut, N. Magesh and C. Abirami. "A comprehensive class of analytic bi-univalent functions by means chebyshev polynomials" *Journal of Fractional Calculus and Applications*, vol. 8, no. 2, pp. 32-39, July 2017.
- [8] H. M. Srivastava, S. S. Eker and R. M. Ali. "Coefficient bounds for a certain class of analytic and bi-univalent functions" *Filomat*, vol. 29, no. 8, pp. 1839-1845, 2015.
- [9] S. K. Lee, V. Ravichandran and S. Supramaniam, "Initial coefficients of bi-univalent functions" *Abstract and Applied Analysis*, vol. 2014, Article ID 640856, 6 pages.
- [10] G. Murugusundaramoorthy, N. Magesh and V. Prameela, "Coefficient bounds for certain subclasses of bi-univalent function" *Abstract and Applied Analysis*, vol. 2013, Article ID 573017, 3 pages.
- [11] E. Deniz, "Certain subclasses of bi-univalent functions satisfying subordinate conditions" *Journal of Classical Analysis*, vol. 2, no. 1, pp. 49-60, 2013.
- [12] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, "Coefficient estimates for bi-univalent Ma-Minda starlike and convex" *Applied Mathematics Letters*, vol. 25, pp. 344-351, 2012.
- [13] B. A. Frasin and M. K. Aouf, "New subclasses of bi-univalent functions" *Applied Mathematics Letters*, vol. 24, pp. 1569-1573, 2011.
- [14] H. M. Srivastava, A. K. Mishra and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions" *Applied Math. Letters*, vol. 23, pp. 1188-1192, 2010.
- [15] O. S. Babu, C. Selvaraj and G. Murugusundaramoorthy, "Subclasses of bi-univalent functions based on Hohlov operator" *International Journal of Pure and Applied Mathematics*, vol. 102, no. 3, pp. 473-482, 2015. [16] J. Jothibasu, "certain subclasses of bi-univalent functions defined by Salagean operator" *Electronic Journal of Mathematical Analysis and Applications*, vol. 3, no 1, pp. 150-157, 2015.
- [17] Z. Peng, G. Murugusundaramoorthy and T. Janani, "Coefficient estimate of bi-univalent functions of complex order associated with Hohlov operator" *Journal of Complex Analysis*, vol. 2014, article ID 693908, 6 pages.
- [18] C. Selvaraj and G. Thirupathi, "Coefficient bounds for a subclass of biunivalent functions using differential operators" *Ann. Acad. Rom. Sci. Ser. Math. Appl.*, vol. 6, no. 2, pp. 204-213, 2014.
- [19] H. M. Srivastava, G. Murugusundaramoorthy and N. Magesh, "Certain subclasses of bi-univalent functions associated with the Hohlov operator" *Global Journal of Mathematical Analysis*, vol. 1, no. 2, pp. 67-73, 2013.
- [20] T. Panigrahi and G. Murugusundaramoorthy, "Coefficient bounds for biunivalent analytic functions associated with Hohlov operator" *Proceedings of the Jangjeon Math. Soc.* Vol. 16, no. 1, pp. 91-100, 2013.
- [21] M. K. Aouf, R. M. El-Ashwah and A. M. Abd-Eltawab, "New subclasses of bi-univalent functions involving Dziok-Srivastava operator" *ISRN Mathematical Analysis*, vol. 2013, article ID 387178, 5 pages.