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Representation of Power Functions in Mathematical Series

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Abstract. In this paper we discuss a problem of generalization of binomial distributed triangle, that is sequence A287326 in OEIS. The main property of A287326 that it returns a perfect cube n as sum of n-th row terms over k, $0 \le k \le n-1$ or $1 \le k \le n$, by means of its symmetry. In this paper we have derived a similar triangles in order to receive powers $m = 5,7$ as row items sum and generalized obtained results in order to receive every odd-powered monomial n^{2m+1}, $m \ge 0$ as sum of row terms of corresponding triangle.

1. Structure of the manuscript

The problem of finding expansions of monomials, binomials, trinomials, etc. is classical and a lot of theorems have been found, the most prominent examples are Binomial Theorem [2], Multinomial theorem, Wozpitsky Identity [30], Stirling numbers of second kind identity, etc. In this paper we try to solve the classical problem of finding expansions of monomials. We start from binomial distributed triangle A287326 [11] in OEIS. The main property of A287326 that it returns a perfect cube n as n-th row sum, starting from $0,\ldots,n-1$ or from $1,\ldots,n$ by means of its symmetry. Therefore, the following question stated:

• Can we find similar to A287326 triangles in order to receive monomial n^t, $t > 3$ as sum of row terms? In other words, can A287326 be generalized in order to receive monomial n^t, $t > 3$ as sum of row terms?

2. Introduction

Let describe the derivation of the sequence A287326 in OEIS. Sequence A287326 returns the perfect cube n as row sum over k, $0 \le k \le n-1$, as well as sum over $1 \le k \le n$, by means of its symmetry. First, consider a difference table of perfect cubes ([4], eq.

Table 1: Difference table of perfect cubes n, $0 \le n \le 10$ up to 3rd order. Reviewing above table, we have noticed that

$$
\begin{aligned}
\Delta(0^3) &= 1+6\cdot 0 = 6\tbinom{1}{2}+\tbinom{1}{0}\\
\Delta(1^3) &= 1+6\cdot 0+6\cdot 1 = 6\tbinom{2}{2}+\tbinom{2}{0}\\
\Delta(2^3) &= 1+6\cdot 0+6\cdot 1+6\cdot 2 = 6\tbinom{3}{2}+\tbinom{3}{0}\\
\Delta(3^3) &= 1+6\cdot 0+6\cdot 1+6\cdot 2+6\cdot 3 = 6\tbinom{4}{2}+\tbinom{4}{0}\\
&\vdots\\
\Delta(n^3) &= 1+6\cdot 0+6\cdot 1+6\cdot 2+\cdots+6\cdot n = 6\tbinom{n+1}{2}+\tbinom{n+1}{0}
\end{aligned} \tag{2.2}
$$

Above difference identity is closely related to Faulhaber's sum of cubes, where

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$n^3 = 6\binom{n+1}{3} + \binom{n+1}{1}$, see ([21], p. 9). Note that $\Delta^2(n^3)$ could be found similarly using above identity $\Delta^2(n^3) = 6\binom{n+1}{3-2} + \binom{n+1}{1-2}$.

Property 2.3. *(Generalized finite difference of power using Faulhaber's formula). Consider the identities, ([21], p. 9).*

$$\begin{cases} n^1 = & \binom{n}{1} \\ n^3 = & 6\binom{n+1}{3} + \binom{n}{1} \\ n^5 = & 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1} \end{cases}$$

We can find the first order finite difference of odd power as decreasing the variable of corresponding binomial coefficients by 1, for example

$$\begin{cases} \Delta n^1 = & \binom{n}{0} \\ \Delta n^3 = & 6\binom{n+1}{2} + \binom{n}{0} \\ \Delta n^5 = & 120\binom{n+2}{4} + 30\binom{n+1}{2} + \binom{n}{0} \end{cases}$$

Continue similarly, we can express each difference of order $t \geq 1$. The coefficients $\{1,6,1,120,30,1\}$ in above identities are generated by

$$V_{n,k} = \frac{1}{r}\sum_{j=0}^{r}(-1)^j\binom{2r}{j}(r-j)^{2n}, \tag{2.4}$$

where $r = n-k+1$, this formula was provided by Peter Luschny in [27]. Therefore, for every odd $t > 0$ and $m \geq 0$, we have

$$\Delta^t n^{2m+1} = \sum_{\substack{0\leq k\leq m \\ l\leq 2(m-k)+1-t \\ l \text{ is even}}} V_{m,k}\binom{n+m-k}{l}, \text{ if } t > 0 \text{ and odd}$$

Let be $m \geq 0$, $t > 1$ and even, then

$$\Delta^t n^{2m+1} = \sum_{\substack{0\leq k\leq m \\ l\leq 2(m-k)+1-t \\ l \text{ is odd}}} V_{m,k}\binom{n+m-k}{l}, \text{ if } t > 1 \text{ and even}$$

Let show finite differences, set $m \geq 1$, $t > 1$, then we have finite difference identity

$$\Delta^t n^{2m} = \sum_{\substack{0\leq k\leq m \\ l\leq 2(m-k)+1-t \\ l \text{ is even}}} \frac{1}{n}V_{m,k}\binom{n+m-k}{l}, \text{ if } t > 0 \text{ and odd}$$

And

$$\Delta t n 2m = \sum_{\substack{0\leq k\leq m \\ l\leq 2(m-k)+1-t \\ l \text{ is odd}}} \frac{1}{n}V_{m,k}\binom{n+m-k}{l}, \text{ if } t > 1 \text{ and even}$$

By the identity $\sum_{k=0}^{n-1}\Delta n^m = n^m$, we have right to represent perfect cube n as

$$n^3 = 6\binom{1}{2} + \binom{1}{0} + 6\binom{2}{2} + \binom{2}{0} + 6\binom{3}{2} + \binom{3}{0} + \cdots + 6\binom{n+1}{2} + \binom{n+1}{0} \tag{2.5}$$

Let rewrite it again and display every binomial coefficient as summation $\binom{n+1}{2} =$

$1 + 2 + \cdots + n$, then $n^3 = (1 + 6 \cdot 0) + (1 + 6 \cdot 0 + 6 \cdot 1) + \cdots + (1 + 6 \cdot 0 + \cdots + 6 \cdot (n - 1))$
Particularizing above expression, we get

$$n^3 = n + (n - 0) \cdot 6 \cdot 0 + (n - 1) \cdot 6 \cdot 1 + \cdots + (n - (n - 1)) \cdot 6 \cdot (n - 1) \tag{2.6}$$

Provided that n is natural. Now we apply a compact sigma notation on (2.6), thus

$$n^3 = n + \sum_{1 \le k \le n} 6k(n - k) \tag{2.7}$$

As sum $\sum_{1 \le k \le n} 6k(n - k)$ consists of n terms, we have right to move n in (2.7) under sigma notation, we get

$$n^3 = \sum_{1 \le k \le n} 6k(n - k) + 1 \tag{2.8}$$

Property 2.9. *(Proof of symmetry). Let be a sets $A(n) := \{1, 2, \ldots, n\}$, $B(n) := \{0, 1, \ldots, n\}$, $C(n) := \{0, 1, \ldots, n - 1\}$, let be expression (2.8) defined as*

$$M(n, C(n)) \stackrel{\text{def}}{=} \sum_{k \in C(n)} 6k(n - k) + 1$$

where x is natural-valued variable and $C(n)$ is iteration set of (2.8), then we have equality

$$M(n, A(n)) = M(n, C(n)) \tag{2.10}$$

Let review and define expression (2.6) as

$$U(n, C(n)) \stackrel{\text{def}}{=} n + 6 \cdot \sum_{k \in C(n)} k(n - k)$$

then

$$U(n, A(n)) = U(n, B(n)) = U(n, C(n)) \tag{2.11}$$

Other words, changing of iteration sets of(2.6) and (2.8) by $A(n)$, $B(n)$, $C(n)$ and $A(n)$, $C(n)$, respectively, doesn't change resulting value for each natural x.

Proof. Let be a plot $y(n,k) = 6k(n - k) + 1$, $k \in \mathrm{R}$, $0 \le k \le 10$, given $n = 10$

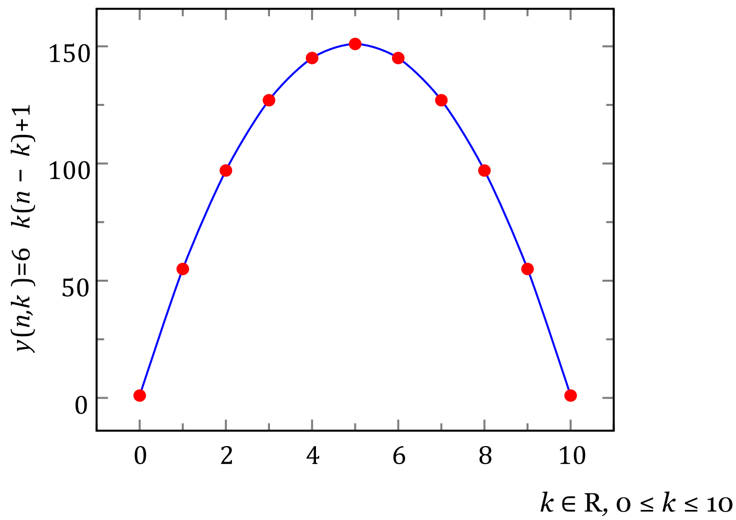

Figure 2. Plot of $6k(n - k) + 1$, $k \in \mathrm{R}$, $0 \leq k \leq n$, where $n = 10$.

Obviously, being a parabolic function, it's symmetrical over $\frac{n}{2}$, hence equivalent $M(n, A(n)) = M(n, C(n))$ follows. Reviewing (2.6) and denote $u(n,k) = kn - k^2$, we can conclude, that $u(n,0) = u(n,n) = 0$, then equality of $U(n, A(n)) =$
$U(n, B(n)) = U(n, C(n))$ immediately follows. This completes the proof.

Review above property (2.9). Let be an example of triangle built using **Definition 2.12.** For every $n \geq 0$

$$D_1(n,k) \overset{\text{def}}{=} 6k(n - k) + 1, \; 0 \leq k \leq n \tag{2.13}$$

over n from 0 to $n = 4$, where n denotes corresponding row and k shows the item of row n.

(2.14)

Row 0: 1
Row 1: 1 1
Row 2: 1 7 1
Row 3: 1 13 13 1
Row 4: 1 19 25 19 1

Figure 3. Triangle generated by $D_1(n,k)$ from 0 to $n = 4$, sequence A287326 in OEIS, [11].

Note that n-th row sum of Triangle (2.14) over $0 \leq k \leq n - 1$ returns perfect cube n. We can see that each row with respect to variable $n = 0, 1, 2, 3, 4, \ldots$, has Binomial distribution of row terms. One could compare Triangle (2.14) with Pascal's triangle [1], [12]

Row 0: 1
Row 1: 1 1
Row 2: 1 2 1

Row 3:	1	3	3	1		
Row 4:	1	4	6	4	1	

Figure 4. Pascal's triangle read by rows, sequence A007318 in OEIS, [1].

Let us approach to show a few properties of triangle (2.14) and $L_1(n,k)$.

Properties 2.15. *Properties of triangle (2.14).*

(1) *Summation of n-th row of triangle (2.14) over k from 0 to n − 1 returns perfect cube n ≥ 0 as follows*

(2.16)
$$n^3 = \sum_{1\le k\le n} D_1(n,k)$$

(2) *First item of each row's number corresponding to central polygonal numbers sequence* $a(n) = \frac{n^2+n+2}{2}$ *(sequence A000124 in OEIS, [13]) returns finite difference of consequent perfect cubes. For example, let be a k-th row of triangle (2.14), such that* $k = \frac{n^2+n+2}{2},\ n = 0,1,2,\cdots$*, then item*

(2.17)
$$\Delta(n^3) = D_1\left(\frac{n^2+n+2}{2},1\right)$$

(3) *Items of (2.14) have Binomial distribution of rows.*

(4) *Linear recurrence, for every k and n > 0*

(2.18)
$$2D_1(n,k) = D_1(n+1,k) + D_1(n-1,k)$$

This linear recurrence is direct result of second order binomial transform of $D_1(n,k)$ by n.

(5) *Linear recurrence, for each n > k*

(2.19)
$$2D_1(n,k) = D_1(2n-k,k) + D_1(2n-k,0)$$

(6) *From (1.24) for every n ≥ 0 follows*

(2.20)
$$n^3 = \sum_{1\le k\le n} D_1(n,k) = \sum_{1\le k\le n} D_1\left(\frac{n^2+n+2}{2},1\right)$$

(7) *Triangle (2.14) is symmetric, i.e*

(2.21)
$$D_1(n,k) = D_1(n,n-k)$$

Property 2.22. *(Generalized binomial series by means of identity (2.16). Let review identity (2.16) in sense of*

$$\sum_{1\le k\le t} D_1(n,k) = \alpha_{0,t}n - \beta_{0,t}$$

By property (2.9) we rewrite above expression as

$$\sum_{0\le k\le t} D_1(n,k) = \alpha_{1,t}n - \beta_{1,t}$$

where subscripts 0,t and 1,t denote the ranges of summation, respectively. Running over t > 0 above identities produce sets of coefficients $\{\alpha_{0,t}\}_t$, $\{\beta_{0,t}\}_t$, $\{\alpha_{1,t}\}_t$ and $\{\beta_{1,t}\}_t$. Below table shows initial terms of these sequences

t	$\alpha_{0,t}$	$\beta_{0,t}$	$\alpha_{1,t}$	$\beta_{1,t}$
	1	0	6	5
	6	4	18	28
	18	27	36	81
	36	80	60	
	60		90	
	90			

Table 5. Array of coefficients $\alpha_{0,1,n}$, $\beta_{0,1,n}$ given $n = 1,\ldots,\overline{10}$. Therefore, perfect cube n could be rewritten as binomials of the form

$$n^3 = \begin{cases} \alpha_{0,n-1}n - \beta_{0,n-1}, & \text{if } t = n-1; \\ \alpha_{1,n}n - \beta_{1,n}, & \text{if } t = n \end{cases}$$

By the main power property, for every $m \in \mathbb{N}$

$$n^m = \begin{cases} \alpha_{0,n-1}n^{m-2} - \beta_{0,n-1}n^{m-3} \\ \alpha_{1,n}n^{m-2} - \beta_{1,n}n^{m-3} \end{cases}$$

We denote above equation as

$$n^m = \alpha_{\overline{0,1},\overline{n-1,n}}n^{m-2} - \beta_{\overline{0,1},\overline{n-1,n}}n^{m-3}$$

Let rewrite the right part of above expression regarding to itself as recursion

$$\begin{aligned} n^m &= \alpha_{\overline{0,1},\overline{n-1,n}}(\alpha_{\overline{0,1},\overline{n-1,n}}n^{m-4} - \beta_{\overline{0,1},\overline{n-1,n}}n^{m-5}) \\ &- \beta_{\overline{0,1},\overline{n-1,n}}(\alpha_{\overline{0,1},\overline{n-1,n}}n^{m-5} - \beta_{\overline{0,1},\overline{n-1,n}}n^{m-6}) \\ &= \alpha^2_{\overline{0,1},\overline{n-1,n}}n^{m-4} - 2\alpha_{\overline{0,1},\overline{n-1,n}}\beta_{\overline{0,1},\overline{n-1,n}}n^{m-5} + \beta^2_{\overline{0,1},\overline{n-1,n}}n^{m-6} \end{aligned}$$

We can observe corresponding binomial coefficient present before each $\alpha_{\overline{0,1},\overline{n-1,n}}$ times $\beta_{\overline{0,1},\overline{n-1,n}}$. Continuous j-times recursion gives

$$n^m = \sum_{k\geq 0}^{\infty}(-1)^k\binom{j}{k}\alpha^{j-k}_{\overline{0,1},\overline{n-1,n}}\beta_{\overline{0,1},\overline{n-1,n}}n^{m-2j-k}, \ j \geq 0$$

Sequences $\alpha_{1,t}$, $\alpha_{0,t>1}$ are generated by $3n^2+3n$, sequence A028896 in OEIS, [23]. *Sequence $\beta_{1,t}$ is generated by $2n^3+3n^2$, sequence A275709 in OEIS,* [20].

In this section we have reached binomial distributed triangle (2.14), such that perfect cube n could be found as sum of n-th row terms of (2.14). Therefore, the follow question is stated

Question 2.23. *Are there exist a similar to A287326 triangles in order to receive monomial n^t, $t > 3$ as sum of n-th row terms, where t is natural? Are there exist a functions $D_v(n,k)$, $v > 1$, such that*

$$n^t \equiv \sum_{1\le k\le n} D_v(n,k),\ v \neq t\ ?$$

3. Generalization of sequence A287326

In order to get analogs of Triangle (2.14) one should solve a system of equations, where unknowns are coefficients of polynomial and variable of polynomial is $k(n-k)$. Let show a triangle generated by $D_2(n,k)$, such that sum of n-th row terms returns n^5.

Example 3.1. We suspect that n-th row of triangle that returns n^5 as sum of n-th row terms is generated by

(3.2) $$D_2(n,k) = A_{2,2}(n-k)^2k^2 + A_{2,1}(n-k)k + A_{2,0},\ n \ge 0,\ 0 \le k \le n,$$

where $A_{2,2}, A_{2,1}, A_{2,0}$ are unknown coefficients. Assume that for every $n \ge 1$ holds

(3.3) $$n^5 \equiv \sum_{1\le k\le n} D_2(n,k)$$

To determine the coefficients $A_{2,2}, A_{2,1}, A_{2,0}$, in (3.2) let rewrite (3.3) in extended view

(3.4) $$A_{2,2}\sum_{1\le k\le n} k^2(n-k)^2 + A_{2,1}\sum_{1\le k\le n} k(n-k) + \sum_{1\le k\le n} A_{2,0}$$

$$= A_{2,2}\sum_{1\le k\le n} k^2(n^2-2nk+k^2) + A_{2,1}\sum_{1\le k\le n} \left(kn-k^2\right) + \sum_{1\le k\le n} A_{2,0}$$

$$= A_{2,2}\sum_{1\le k\le n} \left(k^2n^2-2nk^3+k^4\right) + A_{2,1}\sum_{1\le k\le n} \left(kn-k^2\right) + \sum_{1\le k\le n} A_{2,0}$$

$$= A_{2,2}n^2\sum_{1\le k\le n} k^2 - 2A_{2,2}n\sum_{1\le k\le n} k^3 + A_{2,2}\sum_{1\le k\le n} k^4 + A_{2,1}n\sum_{1\le k\le n} k$$

$$- A_{2,1}\sum_{1\le k\le n} k^2 + \sum_{1\le k\le n} A_{2,0} = n^5$$

Thus, we have received expression containing sums of powers of successive natural numbers, where powers are $\{1,2,3,4\}$. By Faulhaber's formula [7] the following identities hold

(3.5) $$\sum_{1\le k\le n} k = \frac{n^2+n}{2},$$

(3.6) $$\sum_{1\le k\le n} k^2 = \frac{2n^3+3n^2+n}{6},$$

(3.7) $$\sum_{1\le k\le n} k^3 = \frac{n^4+2n^3+n^2}{4},$$

(3.8)
$$\sum_{1\le k\le n} k^4 = \frac{6n^5+15n^4+10n^3-n}{30}.$$

Now we substitute above identities to (3.4), respectively

$$A_{2,2}n^2\frac{2n^3+3n^2+n}{6} - 2A_{2,2}n\frac{n^4+2n^3+n^2}{4} + A_{2,2}\frac{6n^5+15n^4+10n^3-n}{30}$$
$$+ \quad A_{2,1}n\frac{n^2+n}{2} - A_{2,1}\frac{2n^3+3n^2+n}{6} + A_{2,0}n$$

Particularizing the elements of above expression and moving them under the common divisor, we get

(3.9)
$$\frac{A_{2,2}n^5 - A_{2,2}n + 30A_{2,0}}{30} + A_{2,1}\frac{n^3-n}{6}$$

We have to remember that expression (3.9) is the left side of the input equation (3. 3). Therefore,

(3.10)
$$n^5 = \frac{A_{2,2}n^5 - A_{2,2}n + 30A_{2,0}}{30} + A_{2,1}\frac{n^3-n}{6}$$

In order to satisfy (3.10) for each natural n, coefficients $A_{2,0}$,$A_{2,1}$,$A_{2,2}$ should be a solutions of following system of equations

$$\begin{cases} \frac{1}{30}A_{2,2} & = 1 \\ A_{2,1} & = 1 \\ 30A_{2,0} - A_{2,2} & = 0 \end{cases}$$

The only solution of above system is $A_{2,2} = 30$, $A_{2,1} = 0$, $A_{2,0} = 1$. Hereby, $D_2(n,k)$ takes the form

(3.11) $D_2(n,k) = 30k^2(n-k)^2 + 1$

And for every natural $n \ge 1$ holds

(3.12) $n^5 = \sum_{1\le k\le n} D_2(n,k) = \sum_{1\le k\le n} 30k^2(n-k)^2 + 1$

Let show few initial rows of triangle built by $D_2(n,k)$

1

1 1

1 31 1

(3.13) 1 121 121 1

1 271 481 271 1

1 480 1081 1081 481 1

...

Figure 6. Triangle generated by $D_2(n,k)$, $n \ge 0$, $0 \le k \le n$, sequence A300656 in OEIS, [15].

Similarly, finding the coefficients $A_{3,0}$,$A_{3,1}$,$A_{3,2}$,$A_{3,3}$ in

(3.14) $D_3(n,k) = A_{3,3}k^3(n-k)^3 + A_{3,2}k^2(n-k)^2 + A_{3,1}k(n-k) + A_{3,0}$, we get $A_{3,3} = 140, A_{3,2} = -14, A_{3,1} = 0, A_{3,0} = 1$, therefore, for each integer $n \geq 1$ holds

(3.15) $$n^7 = \sum_{1\leq k\leq n} D_3(n,k) = \sum_{1\leq k\leq n} 140k^3(n-k)^3 - 14k^2(n-k)^2 + 1$$

Below we show a few initial rows of triangle built by $D_3(n,k)$

```
                              1
                            1   1
                        1    127    1
(3.16)             1    1093   1093    1
              1    3793    8905    3793    1
          1   8905   30157   30157   8905    1
...
```

Figure 7. Triangle generated by $D_3(n,k)$, $n \geq 0$, $0 \leq k \leq n$, sequence A300785 in OEIS, [16].

We assume now that generalization of A287326 holds for odd powers only. To generalize our sequences A287326, A300656, A300785 for every odd power $2m+1$, $m \geq 0$ one has to review the generating functions of sequences A287326, A300656, A300785 as follows. Let be definition

Definition 3.17.

$D_m(n,k) := A_{m,m}k^m(n-k)^m + A_{m,m-1}k^{m-1}(n-k)^{m-1} + \cdots + A_{m,0}$ where $A_{m,j}$, $0 \leq j \leq m$ are unknown coefficients. Therefore,

(3.18) $$n^{2m+1} = \sum_{1\leq k\leq n} D_m(n,k)$$

We want to notice that it couldn't be applied a compact sigma notation on definition (3.17), i.e we can't write (3.17) as $\sum_{0\leq j\leq m} A_{m,j}k^j(n-k)^j$ since the sum (3.17) returns indeterminate form of $\sum_{0\leq j\leq m} A_{m,j}k^j(n-k)^j$ on step $k = n$ as $A_{m,0}(n-n)^0k^0$. Note that $D_m(n,k)$ is generalization of (2.12) and (3.11). For example, generating functions of sequences A287326, A300656, A300785 are, respectively

$$D_1(n,k) = 1 + 6k(n-k), \text{for } A287326$$

$$D_2(n,k) = 1 - 0k(n-k) + 30k^2(n-k)^2, \text{for } A300656$$

$$D_3(n,k) = 1 - 14k(n-k) + 0k^2(n-k)^2 + 140k^3(n-k)^3, \quad \text{for } A300785$$

Where coefficients $A_{m,j}$, for $m = 1,2,3$ are $\{A_{1,j}\}_{j=0}^{1} = \{1,6\}$, $\{A_{2,j}\}_{j=0}^{2} = \{1,0,30\}$, $\{A_{3,j}\}_{j=0}^{3} = \{1,-14,0,140\}$ in definitions of generating functions of A287326, A300656, A300785. To generalize above result in order to

[1] One can produce a list of solutions of system (2.4) up to $r = 11$ using Mathematica code solutions system 2 4.txt, [24].

receive monomial n^{2m+1} as $\mathrm{P}_{1\leq k\leq n}\, D_m(n,k) = n^{2m+1}$, $m = 0,1,2,...$ one has to solve the system of equations. Complete set of coefficients $\{A_{m,0}, ..., A_{m,m}\}$ such that $\mathrm{P}_{1\leq k\leq n}\, D_m(n,k) = n^{2m+1}$, $m \geq 0$ holds can be found by solving the following system of equations

□$D_m(1,0) = 1_{2m+1}$

□□□□□□□$DD_{mm}(3(2,,0) + 0) + DD_{mm}(3(2,,1) = 21) + D_{2mm}(3_{+1},2) = 3_{2m+1}$

(3.19)

□□□□□□□$...D_m(r,0) + D_m(r,1) + \cdots + D_m(r,r-1) = r^{2m+1}$, $r > m$

List of solutions[1] of system (2.4) is split and assigned to OEIS under the numbers A302971 (numerators of $A_{m,j}$) and A304042 (denominators of $A_{m,j}$). To reach recurrent formula of $A_{m,j}$, first let fix the unused values $A_{m,j} = 0$, for $j < 0$ or $j > m$, so we don't need to care about the summation range for j, then by expanding $(n - k)^j$ and using Faulhaber's formula [7], we get

$$\begin{aligned}
(3.20)\quad \sum_{k=0}^{n-1}(n-k)^j k^j &= \sum_{k=0}^{n-1}\sum_{i}^{\infty}\binom{j}{i}n^{j-i}(-1)^i k^{i+j} \\
&= \sum_{i}^{\infty}\binom{j}{i}n^{j-i}\frac{(-1)^i}{i+j+1}\left[\sum_{t}^{\infty}\binom{i+j+1}{t}B_t n^{i+j+1-t} - B_{i+j+1}\right] \\
&= \underbrace{\sum_{i,t}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t}B_t n^{2j+1-t}}_{(\star)} - \underbrace{\sum_{i}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}B_{i+j+1}n^{j-i}}_{(\diamond)}
\end{aligned}$$

where B_t are Bernoulli numbers [14]. Now, we notice that

$$(3.21)\qquad = 0;\quad \sum_{i}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t} = \begin{cases}\frac{1}{(2j+1)\binom{2j}{j}}, & \text{if } t \\ \frac{(-1)^j}{t}\binom{j}{2j-t+1}, & \text{if } t > 0\end{cases}$$

In particular, the last sum is zero for $0 < t \leq j$. Now we revise the (?) part of (3.20) according to results of (3.21), thus

$$\begin{aligned}
\sum_{i,t}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t}B_t n^{2j+1-t} &= \frac{1}{(2j+1)\binom{2j}{j}} \\
&+ \sum_{t>0}\frac{(-1)^j}{t}\binom{j}{2j-t+1}B_t n^{2j+1-t}
\end{aligned}$$

Therefore, (3.20) takes the form

$$\sum_{k=0}^{n-1}(n-k)^j k^j = \underbrace{\frac{1}{(2j+1)\binom{2j}{j}} + \sum_{t>0}\frac{(-1)^j}{t}\binom{j}{2j-t+1}B_t n^{2j+1-t}}_{(\star)} - \underbrace{\sum_{i}^{\infty}\binom{j}{i}\frac{(-1)^i}{i+j+1}B_{i+j+1}n^{j-i}}_{(\diamond)} \tag{3.22}$$

Now, we keep our attention to (3.22) and we have to remember that if the sum over some variable i contains $\binom{j}{i}$, then instead of limiting its summation range to $i = 0,...,j$, we can let $i = -\infty,...,+\infty$ since $\binom{j}{i} = 0$ for i outside the range $i = 0,...,j$ (i.e., when $i < 0$ or $i > j$). It's much easier to review such sum as summing from $-\infty$ to $+\infty$ (unless specified otherwise), where only a finite number of terms are nonzero, this fact is discussed in [28] as well. To combine or cancel identical terms across the two sums in (3.22) more easily, we introduce ` = $2j+1-t$ to (?) and $\ell = j - i$ to ($\diamond$), respectively, we get

$$\sum_{k=0}^{n-1}(n-k)^j k^j = \frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1} + \sum_{\ell=-\infty}^{\infty}\frac{(-1)^j}{2j+1-\ell}\binom{j}{\ell}B_{2j+1-\ell}n^{\ell} - \sum_{\ell=-\infty}^{\infty}\binom{j}{\ell}\frac{(-1)^{j-\ell}}{2j+1-\ell}B_{2j+1-\ell}n^{\ell} = \frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1} + 2\sum_{\text{odd } \ell}^{\infty}\frac{(-1)^j}{2j+1-\ell}\binom{j}{\ell}B_{2j+1-\ell}n^{\ell}. \tag{3.23}$$

Now, using the definition of $A_{m,j}$, we obtain the following identity for polynomials in n

$$\sum_{j}^{\infty} A_{m,j}\frac{1}{(2j+1)\binom{2j}{j}}n^{2j+1} + 2\sum_{j,\text{ odd } \ell}^{\infty} A_{m,j}\binom{j}{\ell}\frac{(-1)^j}{2j+1-\ell}B_{2j+1-\ell}n^{\ell} \equiv n^{2m+1}. \tag{3.24}$$

Taking the coefficient of n^{2m+1} in above expression, we get $A_{m,m} = (2m+1)\binom{2m}{m}$, and taking the coefficient of x^{2d+1} for an integer d in the range $m/2 \le d < m$ we get $A_{m,d} = 0$. Taking the coefficient of n^{2d+1} in (2.8) for $m/4 \le d < m/2$, we get

$$A_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0 \tag{3.25}$$

, i.e

$$A_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}. \tag{3.26}$$

Continue similarly, we can express $A_{m,j}$ for each integer j in range $m/2^{s+1} \le j < m/2^s$ (iterating consecutively $s = 1,2,...$) via previously determined values of $A_{m,d}$, $d < j$ as follows

$$A_{m,j} = (2j+1)\binom{2j}{j}\sum_{d=2j+1}^{m} A_{m,d}\binom{d}{2j+1}\frac{(-1)^{d-1}}{d-j}B_{2d-2j}. \tag{3.27}$$

The same formula holds also for $m = 0$. Note that in above sum m have to be $m \ge 2j + 1$ to return nonzero term $A_{m,j}$.

Definition 3.28. We define here a generating function of sequence of coefficients $A_{m,j}$, such that $\sum_{k=0}^{n-1} D_m(n,k) = n^{2m+1},\ n \ge 0,\ m \ge 0$, where $D_m(n,k)$ is defined by (3.17)

$$A_{m,j} := \begin{cases} 0, & \text{if } j < 0 \text{ or } j > m \\ (2j+1)\binom{2j}{j}\sum_{d=2j+1}^{m} A_{m,d}\binom{d}{2j+1}\frac{(-1)^{d-1}}{d-j}B_{2d-2j}, & \text{if } 0 \le j < m \\ (2j+1)\binom{2j}{j}, & \text{if } j = m \end{cases}$$

Five initial rows of triangle generated by $A_{m,j}$ are

1

1 6

1 0 30

(3.29) 1 -14 0 140

1 -120 0 0 630 1 -1386 660 0 0 2772

...

Figure 8. Triangle generated by $A_{m,j}$, $j \ge 0$, $0 \le j \le m$, sequences A302971 (numerators of $A_{m,j}$) and A304042 (denominators of $A_{m,j}$).

Note that starting from row $m \ge 11$ the terms of Triangle (3.29) consist fractional numbers, for example, $A_{11,1}$ = 800361655623,6. One can find complete list of the numerators and denominators of $A_{m,j}$ in OEIS under the identifiers A302971 and A304042, respectively, see [17],[18]. To verify the terms that definition (3.28) produces one should refer to Mathematica code[2]. Hereby, let be theorem

Theorem 3.30. *For every positive integers n and m holds*

$$n^{2m+1} = \sum_{1 \le k \le n} D_m(n,k)$$

One can verify results concerning above theorem (3.30) via Mathematica code[3]. Therefore, theorem (3.30) answers to the question question (2.23) positively, since for every $m \ge 0$ exists a triangle, generated by $D_m(n,k)$, $n \ge 0$, $0 \le k \le n$, such that odd power n^{2m+1} can be reached as sum of n-th row of corresponding triangle over $1 \le k \le n$ or $0 \le k \le n-1$. Sequences A287326, A300656, A300785 are partial cases of theorem (3.29) for $m = 1,2,3$, respectively.

3.1. **Properties of $D_m(n,k)$ and $A_{m,j}$.** Here we show a few properties of definition $D_m(n,k)$, some of them correlates with properties of partial case $D_1(n,k)$ in 2.15.

(1) Sum of $A_{m,j}$, defined by (3.27) gives

$$\sum_{j \ge 0} A_{m,j} = 2^{2m+1} - 1$$

(2) Similarly to particular property (1.28), items of $\{D_m(n,k)\}_{k=0}^{n}$, $m \ge 0$, $n \ge 0$, $0 \le k \le n$ is symmetric, i.e

$$D_m(n,k) = D_m(n,n-k)$$

[2] def 2 12.txt, [25].

[3] expression 2 1.txt, [26]. In this code we defined $D_m(n,k) := A_{m,0} + \sum_{j \ge 1} A_{m,j} k^j (n-k)^j$.

(3) From (2) for every $n \geq 1, m \geq 0$ immediately follows

$$\sum_{1\leq k\leq n} D_m(n,k) = \sum_{0\leq k\leq n-1} D_m(n,k)$$

(4) For every $m \geq 0$ the $A_{m,m}$ are terms of the sequence A002457, [19].

(5) For each $m \geq 0$

$$A_{m,0} = 1$$

And

$$\sum_{j\geq 0} A_{m,j} = \sum_{j\geq 0} \binom{2m+1}{j} - 1$$

where $\binom{2m+1}{j}$ is binomial coefficient.

(6) For each even power $2m$, $m \geq 0$ and $n \in \mathbb{N}$, we have

$$n^{2m} = \sum_{1\leq k\leq n} \sum_{j\geq 0} \frac{1}{n} D_m(n,k)$$

3.2. **Example of use.** In this subsection we show a detailed application of theorem (3.30). Recall existing pattern, that is triangle of coefficients $A_{m,j}$ defined by (3.28)

1
1 6
1 0 30
1 -14 0 140
1 -120 0 0 630 1 -1386 660 0 0 2772
...

Figure 9. Triangle generated by $A_{m,j}$, $m \geq 0$, $0 \leq j \leq m$.

By received formula $\sum_{k=1}^{n} \sum_{j\geq 0} D_m(n,k) = n^{2m+1}$ each line of above triangle being multiplied by $T^{j>0}(n,k)$ and summed up to n or $n-1$ over k from 0 or 1, respectively, will result odd power of n^{2m+1}, depending on the row $A_{m,j}$, $0 \leq j \leq m$ is applied. Consider the case $n = 3$, $m = 2$, we introduce triangle built using $T(n,k) = k(n-k)$, $n \geq 1$, $1 \leq k \leq n$ as upper triangular array,

$$T(n,k) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ & 0 & \underline{2} & 4 & 6 \\ & & 0 & 3 & 6 \\ & & & 0 & 4 \\ & & & & 0 \end{pmatrix}, \quad 1 \leq n \leq 5 \tag{3.31}$$

Figure 10. Triangle generated by $T(n,k)$, $n \geq 1$, $1 \leq k \leq n$, sequence A094053, [29] in OEIS.

Then,

$$\begin{aligned} 3^{2\cdot 2+1} &= 1 + 0\cdot 2^1 + 30\cdot 2^2 \\ &+ 1 + 0\cdot \underline{2}^1 + 30\cdot \underline{2}^2 \\ &+ 1 + 0\cdot 0^1 + 30\cdot 0^2 \\ &= 121 + 121 + 1 = 243 \end{aligned}$$

We've highlighted the terms of $A_{2,j}$ and $T(3,k)$ with different colors to be more easily to see regularity. Result we received are terms of the third row of triangle
A300656. Let show another example for $m = 3$, $n = 3$, that is

$2^2 + 140 \cdot 2^3$

$2^2 + 140 \cdot 2^3$

$0^2 + 140 \cdot 0^3$

$1 = 2187$

Consider an example for $m = 3$, $n = 4$, we have

$1 - 14 \cdot 3^1 + 0 \cdot 3^2 + 140 \cdot 3^3$

$1 - 14 \cdot 4^1 + 0 \cdot 4^2 + 140 \cdot 4^3$

$1 - 14 \cdot 3^1 + 0 \cdot 3^2 + 140 \cdot 3^3$

$1 - 14 \cdot 0^1 + 0 \cdot 0^2 + 140 \cdot 0^3$

$3739 + 8905 + 3739 + 1 = 16384$

= 5

= $4^2 + 140 \cdot 4^3$

$6^2 + 140 \cdot 6^3$

$6^2 + 140 \cdot 6^3$

$4^2 + 140 \cdot 4^3$

$0^2 + 140 \cdot 0^3$

$30157 + 8905 + 1 = 78125$

For $m = 4$, $n = 5$ we have

$\cdot 4^2 + 0 \cdot 4^3 + 630 \cdot 4^4$

$\cdot 6^2 + 0 \cdot 6^3 + 630 \cdot 6^4$

$\cdot 6^2 + 0 \cdot 6^3 + 630 \cdot 6^4$

$\cdot 4^2 + 0 \cdot 4^3 + 630 \cdot 4^4$

$\cdot 0^2 + 0 \cdot 0^3 + 630 \cdot 0^4$

$61 + 815761 + 160801 + 1 = 1953125$

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5. Conclusion

In this paper particular pattern, that is binomial distributed triangle A287326 in OEIS, which shows perfect cube n as sum of row terms over $0 \le k \le n-1$ or $1 \le k \le n$ is generalized. Firstly, we discussed analogs of A287326 for powers $2m+1 = 5,7$, sequences A300656, A300785, respectively, then we derived coefficients $A_{m,j}$, such that for every $n \ge 0$ and $m \ge 0$ holds

$$\sum_{1\le k\le n}\sum_{j\ge 0} A_{m,j} T^j(n,k) = n^{2m+1}$$

where $A_{m,j}$ is defined by definition (3.27). Therefore, question question (2.23) is answered positively. Section 3 is totally dedicated to complete and extended derivation of identity $\sum_{1\le k\le n}\sum_{j\ge 0} A_{m,j}T^j(n,k) = n^{2m+1}$. Properties of triangle (2.14) and $L_m(n,k)$ are shown in properties 2.15 and subsection 3.1, respectively. Relation between Faulhaber's sum $\sum n^m$ and finite differences of power are shown in 2.3.

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