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Symmetry-Enhanced Computational Strategies for Multiple Root Identification

Rohan Chandra, Aarav Goyal, and Kaustubh Mehta
Department of Mathematics, Amritsar University, Amritsar 143001, India

Abstract: There are a few optimal eighth order methods in literature for computing multiple zeros of a nonlinear function. Therefore, in this work our main focus is on developing a new family of optimal eighth order iterative methods for multiple zeros. The applicability of proposed methods is demonstrated on some real life and academic problems that illustrate the efficient convergence behavior. It is shown that the newly developed schemes are able to compete with other methods in terms of numerical error, convergence and computational time. Stability is also demonstrated by means of a pictorial tool, namely, basins of attraction that have the fractal-like shapes along the borders through which basins are symmetric.

1. Introduction

Finding out the roots of nonlinear equations is an important task in numerical mathematics and has many advantages in engineering and applied sciences [1–5]. In the present work, we consider numerical methods for locating the multiple root α of multiplicity m of a nonlinear equation $f(x) = 0$. Mathematically, by multiple root α of multiplicity m, we mean that $f^{(j)}(\alpha) = 0, j = 0, 1, 2, \ldots, m-1$ and $f^{(m)}(\alpha) \neq 0$.

Based on the quadratically convergent modified Newton's method (see [6])

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

many higher order methods have been developed in literature. For example, see [7–18] and references shown therein. According to Traub's terminology (see [5]), the Newton's method (1) is a one-point optimal order method. In recent years, a number of two-point optimal fourth order methods have been proposed for multiple zeros (see [10–12,14–16,19–25]). Some non-optimal multipoint methods of sixth order are developed in [17,26]. More recently, multipoint methods with optimal eighth order convergence have also been proposed in the literature which are shown below:

Behl et al. [27] derived a three-point family of optimal method with eighth-order convergence, which is given by

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$yn = xn - m \frac{f(x\ )}{f\ (x_n)}$ n

$z_n = x_n - \ n \frac{f(x_n)}{f\ (x_n)},\quad u_nQ(u\ )$

$f(x\ )$ n

$$xn+1 = zn - untnG(hn, tn) f\ o(xn), \tag{2}$$

$\frac{1}{\ }m = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}} = \frac{u}{a_1+}$ where u_n, $\left(\frac{f(z_n)}{f(y_n)}\right)$ complex $h_n\ a^n 2un$ and $t_n =$ ___ , being a_1 and a_2 non-zero free parameters, $Q : C \to C$ and $Q : C^2 \to C$ is a holomorphic function in the neighborhood of origin (0) and (0, 0).

Zafar et al. [28] proposed an optimal iterative scheme with eighth-order convergence, which is given as follows:

$$yn = xn - m \frac{f(x}{f} 0(xnn))$$

$$zn = xn - munH(un) \frac{f(x}{f} 0(xnn)),$$

$$xn+1 = zn - untn(B1 + B2un)P(tn)G(wn) \frac{f(x}{f} 0(xnn)), \tag{3}$$

where $B_1, B_2 \in R$ are free parameters and the weight functions $H : C \to C$, $P : C \to C$ and $G : C \to C$

are analytic functions in the neighborhood of origin (0) with $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $t_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$ and

$wn = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$.

Geum et al. [29] have derived an optimal iterative scheme with eighth order convergence with weight function approach as follows:

$$yn = xn - m\, ff0((xxnn))$$

$$wn = xn - mLf\,(s)\, ff0((xxnn)),$$

$$xn+1 = xn - m\, Lf \frac{f(x}{f\,(xn)} (s) + K_f(s,u))\ ,\ n)\ , \tag{4}$$

where weight functions $L_f : C \to C$ and $K_f : C^2 \to C$ are holomorphic functions in the neighborhood

of origin (0) and (0, 0) with $s = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$ and $u = \left(\frac{f(w_n)}{f(y_n)}\right)^{\frac{1}{m}}$.

Behl et al. [30] also developed another three-point optimal iterative scheme with eighth order convergence involving free parameters for multiple zeros using a weight function approach as follows:

$$\frac{f(x_n)}{f'(x_n)}$$

$$yn = xn - m \quad \frac{f(x_n)}{} \quad + \beta u_n$$

$$wn = yn - mu f'(xn)\, 1 + (\beta - 2)u, \beta \in R$$

$$x_{n+1} = zn - uv \frac{f(x_n)}{f'(xn)} \left(\alpha_1 + (1 + \alpha_2 v)P_f(u)\right), \quad (5)$$

where $\alpha_1, \alpha_2 \in$ R are free parameters and the weight function P_f: C → C is an analytic function in the

$\frac{1}{m}$ neighborhood of origin $= \left(\frac{f(y_n)}{f(x_n)}\right)$ (0) $= \left(\frac{f(w_n)}{f(y_n)}\right)^{\frac{1}{m}}$ with u and v.

Behl et al. [31] proposed yet another three-point optimal iterative scheme with eighth-order convergence involving free parameters for multiple zeros which is as follows:

$$yn = xn - m \frac{f(xn)}{f'(xn)}$$

$$wn = yn - munGf(un) \frac{f(xn)}{f'(xn)},$$

$$xn+1 = zn + \frac{u_n w_n}{1-w_n} \frac{f(xn)}{f'(xn)} \left(Hf(un) + Kf(vn)\right), \quad (6)$$

where weight function G_f, H_f, K_f: C → C is a holomorphic function in the neighborhood of origin

(0) with $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $v_n = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$ and $v_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$.

Zafar et al. in [32] also developed a three-point optimal iterative scheme with eighth order convergence involving free parameters as follows:

$$yn = xn - m \frac{f(xn)}{f'(xn)}$$

$$wn = yn - munG_f(un) \frac{f(x_n)}{f'(x_n)},$$

$$xn+1 = zn - munH(un, tn, wn) \frac{f(x_n)}{f'(xn)}, \quad (7)$$

where weight functions G : C → C and H : C^3 → C are holomorphic functions in the neighborhood

$\frac{1}{m}$ of origin (0) and (0, 0, $= \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$ $= \left(\frac{f(z_n)}{f(y_n)}\right)$ 0) $= \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$ with u_n, t_n and w_n.

We are motivated by the research moving in the direction of developing optimal higher order methods. So, we attempt to propose a new class of iterative methods with optimal eighth order for computing multiple zeros. The methodology is based on weight function approach for the construction of a new scheme. Each member of the proposed scheme has optimal order in the sense of the classical Kung–Traub conjecture [5]. The efficiency and robustness of the proposed methods are demonstrated by performing several numerical problems. We observe that our methods have better results than those obtained by the existing methods.

The rest of the paper is summarized as follows. In Section 2, the scheme of eighth order method is developed and its convergence is studied. Some special cases of the family are explored in Section 3. Numerical experiments for several examples are performed in Section 4 to demonstrate the applicability and efficiency of the presented methods. Section 5 contains complex geometry based on the geometrical tool basins of attraction. Concluding remarks are given in Section 6.

2. Construction of the Method

This section contains the construction and convergence analysis of the proposed method with the main theorem. So, we consider the following three-step scheme whose first step is modified Newton iteration (1) for the known multiplicity $m \geq 1$:

$$x_{n+1} = \begin{cases} y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - m u(1 + 2u - u^2) \frac{f(x_n)}{f'(x_n)}, \end{cases} \quad (x\,) \quad (x_n) \quad (x_n), \tag{8}$$

$\frac{1}{}m$: C → $= \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}, v = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}} = \left(\frac{f(z_n)}{f(y_n)}\right)$ where u, w and the functions H : C → C and G C are analytic in the neighborhood of '0'.

In order to discuss the convergence analysis of iterative scheme, (8) the following theorem is proved:

Theorem 1. *Let f : C → C be an analytic function in a domain containing a multiple zero (say, α) having multiplicity m. Assume that starter x_0 is close enough to α, then the iteration scheme expressed by (8) possesses eighth order of convergence, when the following conditions are satisfied*

$$\begin{cases} G(0) = 1, G'(0) = 0, \textit{ and } G''(0) = -12, H(0) \\ = 1, \quad \textit{and} \quad H'(0) = 4. \end{cases}$$

Proof. Let $e_n = x_n - \alpha$ be the error at n-th stage. Expanding $f(x_n)$ about α by Taylor's expansion, we have that

$$f(x_n) = \frac{f^{(m)}(\alpha)e_n^m}{m!}\left(1 + C_1 e_n + C_2 e_n^2 + C_3 e_n^3 + C_4 e_n^4 + C_5 e_n^5 + C_6 e_n^6 + C_7 e_n^7 + C_8 e_n^8 + O(e_n^9)\right) \tag{9}$$

and

$$f'(x_n) = \frac{f^{(m)}(\alpha)e_n^{m-1}}{m!}\left(m + C_1(m+1)e_n + C_2(m+2)e_n^2 + C_3(m+3)e_n^3 + C_4(m+4)e_n^4 + C_5(m+5)e_n^5 + C_6(m+6)e_n^6 + C_7(m+7)e_n^7 + C_8(m+8)e_n^8 + O(e_n^9)\right), \tag{10}$$

where $C_k = \frac{m!\, f^{(m+k)}(\alpha)}{(m+k)!\, f^{(m)}(\alpha)}$ for $k \in \mathbb{N}$.

By using (9) and (10), we obtain that

$$y_n - \alpha = \frac{C_1}{m} e_n^2 + \sum_{i=1}^{6} \omega_i e_n^{i+2} + O(e_n^9), \quad (11)$$

where $\omega_i = \omega_i(m, C_1, C_2, \ldots, C_8)$ are given in terms of $m, C_1, C_2 \ldots C_8$ with explicitly written two coefficients $\omega_1 = \frac{2mC_2-(m+1)C_1^2}{m^2}$, $\omega_2 = \frac{1}{m^3}\{3m^2C_3 + (m+1)^2C_1^3 - m(4+3m)C_1C_2$, etc.

Developing $f(y_n)$ about α,

$$f(y_n) = \frac{f^{(m)}(\alpha)}{m!}\left(\frac{C_1}{m}\right)^m e_n^{2m}\left(1 + \frac{2C_2m - C_1^2(m+1)}{C_1} e_n + \frac{1}{2mC_1^2}\{(3+3m+3m^2+m^3)C_1^4 - 2m(2+3m+2m^2)C_1^2C_2 + 4(-1+m)m^2C_2^2 + 6m^2C_1C_3\}e_n^2 + \sum_{i=0}^{4} \bar{\omega}_i e_n^{i+4} + O(e_n^9)\right). \quad (12)$$

By using (9) and (12), we get expression of u as

$$u = \frac{C_1}{m} e_n + \frac{2C_2m - C_1^2(m+2)}{m^2} e_n^2 + \sum_{i=0} \eta_i e_n^{i+3} + O(e_n^7), \quad (13)$$

where $\eta_i = \eta_i(m, C_1, C_2, \ldots, C_8)$ are given in terms of $m, C_1, C_2, \ldots, C_8$ with explicitly written one coefficient $\eta_1 = \frac{1}{2m^3}\{C_1^3(2m^2+7m+7) + 6C_3m^2 - 2C_2C_1m(3m+7)\}$ etc.

Inserting the expressions (9), (10) and (13) in the second step of scheme (8), we obtain that

$$z_n - \alpha = \frac{(11+m)C_1^3 - 2mC_2C_3}{2m^4} e_n^4 + \sum_{j=1}^{3} P_j e_n^{j+4} + O(e_n^9), \quad (14)$$

where $P_j = P_j(m, C_1, C_2, \ldots, C_8)$ and $j = 1, 2, 3, 4$.

Expansion of $f(z_n)$ about α leads us to the expression

$$f(z_n) = \frac{f^{(m)}(\alpha) z_n^m}{m!}\left(1 + C_1 z_n + C_2 z_n^2 + O(z_n^3)\right). \quad (15)$$

Using (9), (12) and (15), we get expressions of v and w

$$v = \frac{(11+m)C_1^3 - 2mC_2C_3}{2m^3} e_n^3 + \sum_{j=0}^{3} Q_j e_n^{j+4} + O(e_n^8), \quad (16)$$

where $Q_j = Q_j(m, C_1, C_2, \ldots, C_7)$, $1 \le j \le 4$

and

$$w = \frac{(11+m)C_1^2 - 2mC_2}{2m^3} e_n^2 + \sum_{j=0}^{4} R_j e_n^{j+3} + O(e_n^8), \quad (17)$$

where $R_j = R_j(m, C_1, C_2, \ldots, C_7)$ are given in terms of $m, C_1, C_2 \ldots C_7$.

Expansions of weight functions $G(u)$ and $H(v)$ in the neighborhood of origin '0' by Taylor series yield

$$G(u) \approx G(0) + uG'(0) + \frac{1}{2}u^2G''(0) \tag{18}$$

and

$$H(v) \approx H(0) + vH'(0) + \frac{1}{2}v^2H''(0). \tag{19}$$

Hence by substituting (9), (10), (13), (16), (18) and (19) into the last step of scheme (8), we obtain the error equation

$$e_{n+1} = \frac{C_1\left(mC_1 - (11+m)C_1^2\right)(-1+H(0))}{2m} e_n^4 + \sum_{i=1}^{3} \Gamma_i e_n^{i+4} + O(e_n^9), \tag{20}$$

where $\Gamma_i = \Gamma_i(m, H(0), C_1, C_2, \ldots, C_7)$ are given in terms of m, $H(0)$, $C_1, C_2 \ldots C_7$ with $i = 1, 2, 3$.

From Equation (20), it is clear that we will obtain at least fifth order convergence when $H(0) = 1$.

Using the value $H(0) = 1$ in $\Gamma_1 = 0$, we will obtain

$$G(0) = 1. \tag{21}$$

By inserting the expression $H(0) = 1$ and $G(0) = 1$ in $\Gamma_2 = 0$, we have

$$G''(0) = 0. \tag{22}$$

Now, with the help of the above independent expressions $H(0) = 1$, $G(0) = 1$ and $G''(0) = 0$ in $\Gamma_3 = 0$, we ascertain that

$$H''(0) = 4. \tag{23}$$

Substituting $H(0) = 1$, (21), (22) and (23) in Equation (20), the error equation showing eighth order convergence is given by

$$\begin{aligned} e_{n+1} = \frac{1}{720m^7}\Big(&\big(655751 + 990429m + 809345m^2 + 335415m^3 + 61304m^4 + 2556m^5\big)C_1^7 - 6m\big(330143 \\ &+ 491815m + 278255m^2 + 61385m^3 + 2882m^4\big)C_1^5C_2 + 30m^2\big(28649 + 29546m + 8971m^2 \\ &+ 514m^3\big)C_1C_3 + 60m^2C_1^3\big(\big(34857 + 35162m + 10263m^2 + 562m^3\big)C_2^2 - 2m\big(2375 + 1320 \\ &+ 103m^2\big)C_4\big) - 360m^3C_1^2\big(\big(3341 + 1740m + 127m^2\big)C_2C_3 - m(167 + 23m)C_5\big) + 360m^4 \\ &\times \big(\big(417 + 43m\big)C_2^3C_3 - 34mC_3C_4 - 26mC_2C_5\big) - 120m^3C_1\big(\big(4247 + 2070m + 139m^2\big)C_2^3 \\ &- 6m(273 + 35m)C_2C_4 - 3m\big((305 + 39m)C_3^2 - 10mC_6\big)\big)\Big)e_n^8 + O(e_n^9). \end{aligned} \tag{24}$$

Thus, proof of theorem is established. □

3. Some Special Cases of Weight Functions of G(u) and H(v)

We explore some special cases of our proposed method (8) by employing different forms of weight functions. In this regard, the following simple members of the proposed family are defined:

Case 1. *Let us describe the following weight functions satisfying the conditions established in Theorem 1:*

$1 + 6u$

$$G(u) = 1\_\_\_\_\_ + 6u + 6u_2 \qquad and \qquad H(v) = 1 + 4v.$$

Then corresponding optimal eighth order iterative scheme is given by

$$f(xn)$$

$$yn = xn - m\, f0(xn),$$

$$zn = yn - mu(1 + 2u - u2)\, ff0((xxnn)), \qquad (25)$$

$$xn{+}1 = zn - m(1 + u)(1 + 4v)v\, ff0((xxnn)) - m(u + w)v1{+}16{+}u{+}6u6u2\, ff0((xxnn)),$$

Case 2. *Let us describe the following weight functions that satisfy the conditions of Theorem 2.1:*

$$1 + 6u \qquad G(u) = \_\_\_\_\__2 \quad and \quad H(v) = \frac{1}{1 - 4v}. \qquad n \quad n \quad f0(xn) \quad zn$$

$$1 + 6u + 6u \qquad = yn - mu(1 + 2u - u2)\, ff0((xxnn)), \qquad (26)$$

Then, the eighth order iterative scheme is

$$f(xn)\; y = x - m\,,$$

$$x_{n+1} = z_n - m\frac{1+u}{1-4v} v \frac{f(x_n)}{f'(x_n)} - m(u+w) v \frac{1+6u}{1+6u+6u^2} \frac{f}{f'}((xx_n n)),$$

Case 3. *Next, the following weight functions from the conditions of Theorem 2.1 are selected:*

$$G(u) = 1\frac{1 + 6u}{\_\_\_\_\_} + 6u + 6u_2 \qquad and \qquad H(v) = 1\frac{1 + 2v}{\_\_\_\_\_} - 2v.$$

For these functions, the corresponding eighth order method is

$$f(xn)$$

$$yn = xn - m\, f0(xn),$$

$$zn = yn - mu(1 + 2u - u^2)\frac{f(xn)}{f0(xn)}, \qquad (27)$$

$$x_{n+1} = z_n - m(1{+}u1{-})(21v{+}2v)v\, ff0((xxnn)) - m(u + w)v1{+}16{+}u{+}6u6u2\, ff0((xxnn)),$$

4. Numerical Results

This section is dedicated to test the efficiency and convergence of the proposed class. To do this, we consider the special cases of the proposed class, namely methods (25)–(27), denoted by NM1, NM2, and NM3, respectively. A total number of four test problems are selected for numerical testing. In addition, we want to compare our methods with other existing robust schemes of eighth order for multiple zeros given by Behl et al. [27,31] and Zafar et al. [28]. In this regard, we consider method (2) for $a_1 = 1$, $a_2 = 1$ by Behl et al. [27], two special cases of method (6) by Behl et al. [31] and two special cases method (7) (for $A_2 = 1$, $P_0 = 1$) by Zafar et al. [28], and denote them by BM1, BM2, BM3, FM1 and FM2, respectively.

Various problems considered for numerical testing are shown in Table 1. Calculations are performed in the Mathematica software using multiple-precision arithmetic. The computed numerical values shown in Tables 2–5 include: (i) number of iterations (n) that are required to find the solution with stopping criterion $|x_{n+1} - x_n| + |f(x_n)| < 10^{-350}$ (ii) values of the last three successive errors $|x_{n+1} - x_n|$, (iii) residual error $f(x_n)$, (iv) computational order of convergence (COC) and (v) elapsed time (CPU—time in seconds) in execution of a program, which is measured by the command

"TimeUsed[ ]". The computational order of convergence (COC) is calculated by applying the formula (see [33])

$$\text{COC} = \frac{\ln|(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln|(x_n - \alpha)/(x_{n-1} - \alpha)|}.$$

Table 1. Test examples.

Example	**Root** (α)	**Multiplicity** (m)	**Initial Guess** (x_0)
Example 1: Standard nonlinear function [26]: $f_1(x) = (x\log x - \sqrt{x} + x^3)^3$	1.000		1.50
Example 2: Standard nonlinear function [17]: $f_2(x) = (xe^{x^2} - \sin^2 x + 3\cos x + 5)^4$	−1.21596937...		−1.50
Example 3: $)(5 - x - x^4 - \sin^2 x)$ **Standard nonlinear function** [26]: $f_3(x) = 9 - 2x - 2x^4 + \cos 2x$	1.29179850...		1.50
Example 4: Eigen value problem [18]: $f_4(x) = x^7 - 17x^6 + 116x^5 - 410x^4 + 809x^3 - 893x^2 + 514x - 120$			0.50

Table 2. Comparison of performance of methods for example 1.

Methods	***n***	$\|e_{n-2}\|$	$\|e_{n-1}\|$	$\|e_n\|$	$f(x_n)$	**COC**	**CPU-Time**
BM1		6.33×10^{-03}	7.63×10^{-16}	3.69 10−119	×5.90 10−2834	×7.9957	0.078
BM2		1.40×10^{-03}	1.20×10^{-21}	1.10 10−062	×2.79 10−0554	×2.2710	0.140
BM3		2.05×10^{-03}	4.23×10^{-21}	4.89 10−061	×1.81 10−0539	×2.2582	0.094
FM1		4.34×10^{-03}	4.36×10^{-17}	4.93 10−129	×9.54 10−3071	×7.9964	0.078
FM2		4.24×10^{-03}	3.64×10^{-17}	1.15 10−129	×6.83 10−3086	×7.9965	0.078

NM1	4.51×10^{-03}	3.05×10^{-17}	1.40×10^{-130}	9.34×10^{-3109}	7.9969	0.093
NM2	3.15×10^{-03}	1.74×10^{-18}	1.60×10^{-140}	2.35×10^{-3347}	7.9980	0.094
NM2	3.91×10^{-03}	9.80×10^{-18}	1.62×10^{-134}	2.96×10^{-3203}	7.9974	0.094

Table 3. Comparison of performance of methods for example 2.

Methods *n*	$\lvert e_{n-2} \rvert$	$\lvert e_{n-1} \rvert$	$\lvert e_n \rvert$	$f(x_n)$	COC	CPU-Time
BM1	8.32×10^{-03}	5.37×10^{-14}	1.68×10^{-103}	1.03×10^{-3270}	7.9947	0.360
BM2	4.34×10^{-04}	1.47×10^{-09}	5.79×10^{-026}	2.57×10^{-0293}	3.0000	0.344
BM3	2.14×10^{-03}	2.15×10^{-19}	1.80×10^{-055}	2.12×10^{-0647}	2.2550	0.344
FM1	5.65×10^{-03}	3.48×10^{-15}	7.80×10^{-113}	1.04×10^{-3568}	7.9947	0.360
FM2	5.53×10^{-03}	2.93×10^{-15}	1.98×10^{-113}	9.76×10^{-3588}	7.9948	0.344
NM1	5.51×10^{-03}	8.72×10^{-16}	3.53×10^{-118}	6.94×10^{-3742}	7.9968	0.343
NM2	2.02×10^{-03}	2.92×10^{-19}	5.64×10^{-146}	2.28×10^{-4631}	7.9990	0.328
NM2	4.13×10^{-03}	8.67×10^{-17}	3.37×10^{-126}	1.64×10^{-3998}	7.9977	0.360

Table 4. Comparison of performance of methods for example 3.

Methods *n*	$\lvert e_{n-2} \rvert$	$\lvert e_{n-1} \rvert$	$\lvert e_n \rvert$	$f(x_n)$	COC	CPU-Time
BM1	6.52×10^{-05}	2.95×10^{-32}	5.20×10^{-251}	4.85×10^{-3999}	7.9999	0.250
BM2	6.81×10^{-09}	2.06×10^{-24}	5.69×10^{-71}	2.96×10^{-418}	3.0000	0.219
BM3	7.63×10^{-06}	1.35×10^{-40}	1.61×10^{-119}	1.53×10^{-709}	2.2711	0.219
FM1	4.88×10^{-05}	3.85×10^{-33}	5.74×10^{-258}	1.10×10^{-4085}	7.9999	0.187
FM2	4.81×10^{-05}	3.40×10^{-33}	2.13×10^{-258}		7.9999	0.203
NM1	3.59×10^{-05}	1.29×10^{-34}	3.67×10^{-270}		7.9999	0.188
NM2	2.73×10^{-05}	1.44×10^{-35}	8.58×10^{-278}		7.9999	0.203
NM2	3.17×10^{-05}	4.79×10^{-35}	1.31×10^{-273}		7.9999	0.204

Table 5. Comparison of performance of methods for example 4.

Methods n	$\|e_{n-2}\|$	$\|e_{n-1}\|$	$\|e_n\|$	$f(x_n)$	COC	CPU-Time
BM1	1.27×10^{-03}	1.47×10^{-22}	4.85×10^{-147}	7.90×10^{-4155}	7.9995	0.078
BM2	2.13×10^{-04}	1.91×10^{-29}	2.02×10^{-086}	3.25×10^{-0769}	2.2747	0.172
BM3	3.30×10^{-04}	5.27×10^{-29}	4.24×10^{-085}	2.55×10^{-0757}	2.2621	0.157
FM1	8.52×10^{-04}	6.89×10^{-24}	1.28×10^{-184}	1.48×10^{-4408}	7.9996	0.109
FM2	8.35×10^{-04}	5.89×10^{-24}	3.65×10^{-185}	1.23×10^{-4421}	7.9996	0.093
NM1	8.50×10^{-04}	3.78×10^{-24}	5.84×10^{-187}	1.59×10^{-4465}	7.9997	0.109
NM2	6.65×10^{-04}	5.32×10^{-25}	8.92×10^{-194}	4.20×10^{-4629}	7.9997	0.109
NM2	7.63×10^{-04}	1.60×10^{-24}	5.87×10^{-190}	1.81×10^{-4537}	7.9997	0.125

From the above tables, we observe that the accuracy is increasing in the values of successive approximations, which points to the good convergence of the methods. The present methods also show consistent convergence behavior as compared to the existing methods. At the time when stopping criterion $|x_{n+1} - x_n| + |f(x_n)| < 10^{-350}$ is attained, the value ' 0 ' for $|x_{n+1} - x_n|$ is displayed. Computational order of convergence shown in the penultimate column of each table overwhelmingly supports the theoretical convergence of order eight. The CPU-time values in the last column of each table show that the new methods utilize less execution time than the time used by existing methods,

which confirms the effectiveness of the proposed techniques. The main motive to apply these methods on different types of nonlinear equations is to illustrate the exactness of the obtained approximate solution and the convergence to the solution. Similar numerical tests, performed on a variety of numerical problems of different kinds, ensured the above remarks to a large extent.

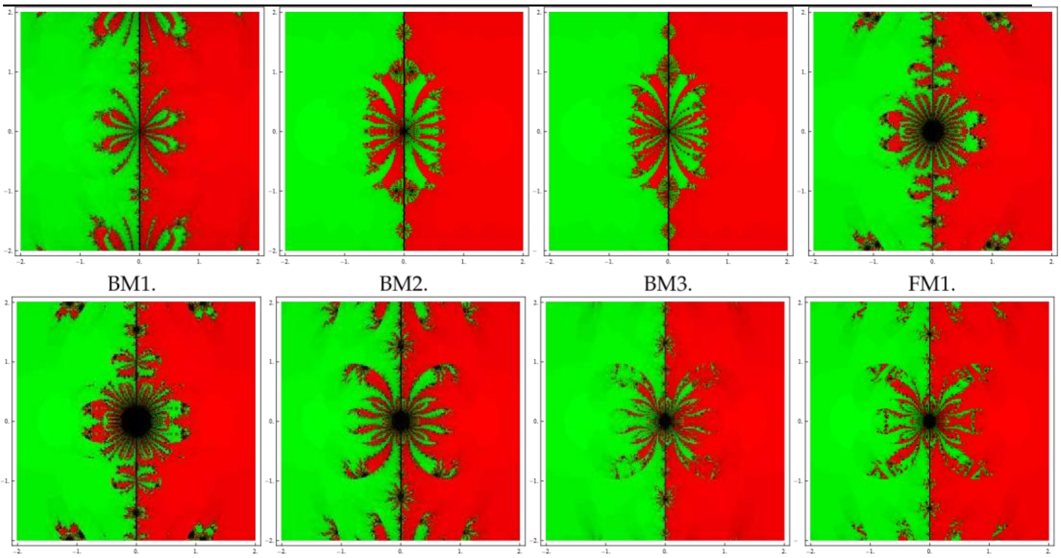
5. Basins of Attraction

We aim to present the complex dynamical nature of new methods based on the geometrical tool, namely basins of attraction of the multiple zeros of a polynomial $P(z)$. Study of basins of attraction provides an important information about the stability and convergence of numerical methods. Initially, this idea was floated by Vrscay and Gilbert [34]. In recent times, many authors have used this idea in their work, see, for example [35,36] and references given there. The basic definitions related to dynamical concepts of rational function associated with iterative methods can be found in [34].

To view the geometry in complex plane, we assess the attraction basins of the roots by applying the methods on some polynomials (see Table 6). The basins of attraction assessed are shown in Figures 1–3 for the considered polynomials. To plot basins we use rectangles $R \in \mathbb{C}$ of size $[-2, 2] \times [-2, 2]$ and $[-3, 3] \times [-3, 3]$, and assign different colors to the basins. Black color is assigned to the points for which the method is divergent.

Table 6. Comparison of performance based on basins of attraction of methods.

S. No.	Test Problems	*m*	Roots	Color of Fractal	Best Performer	Poor Performer
	$P_1(z) = (z^2 - 1)^3$		−1	green	BM1, NM2, NM3,	FM1, FM2
				red	BM2, BM3, NM1	
	$P_2(z) = (z^3 - z)^3$		−1	red	BM3, BM2, NM2	FM1, FM2
				green	NM1, BM1	
				blue		
	$P_3(z) = z^4 - 6z^2 + 8$		−2	red	BM3, NM2, BM2	FM1, FM2
			−1.414	green	NM3, NM1, BM1	
			1.414	yellow		
				blue		



FM2. NM1. NM2. NM3.

Figure 1. Basins of attraction of *BM*1, *BM*2, *BM*3, *FM*1, *FM*2, *NM*1, *NM*2, *NM*3 for polynomial $P_1(z)$.

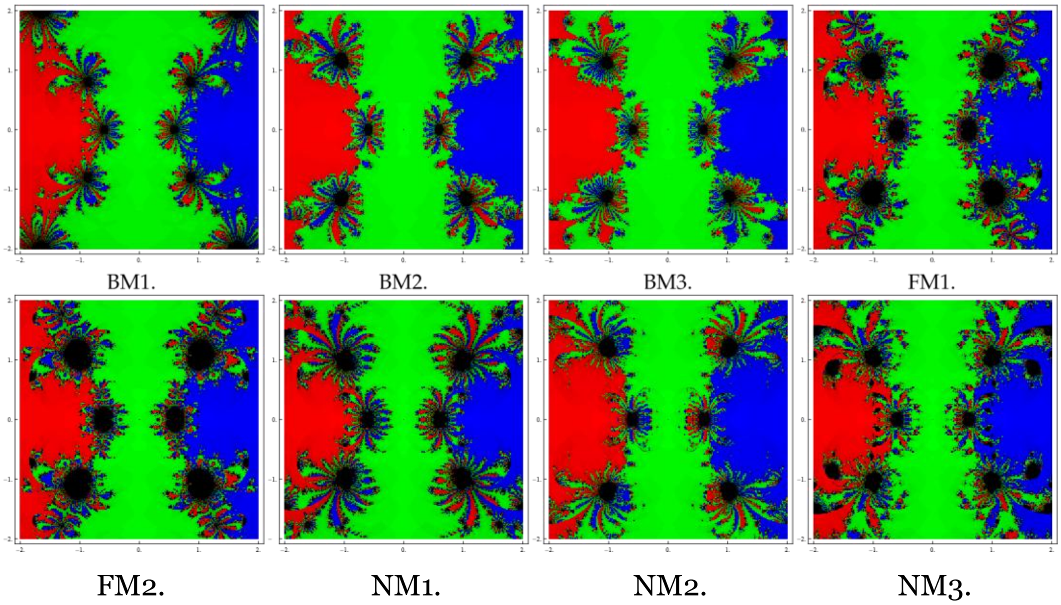


Figure 2. Basins of attraction of *BM*1, *BM*2, *BM*3, *FM*1, *FM*2, *NM*1, *NM*2, *NM*3 for polynomial $P_2(z)$.

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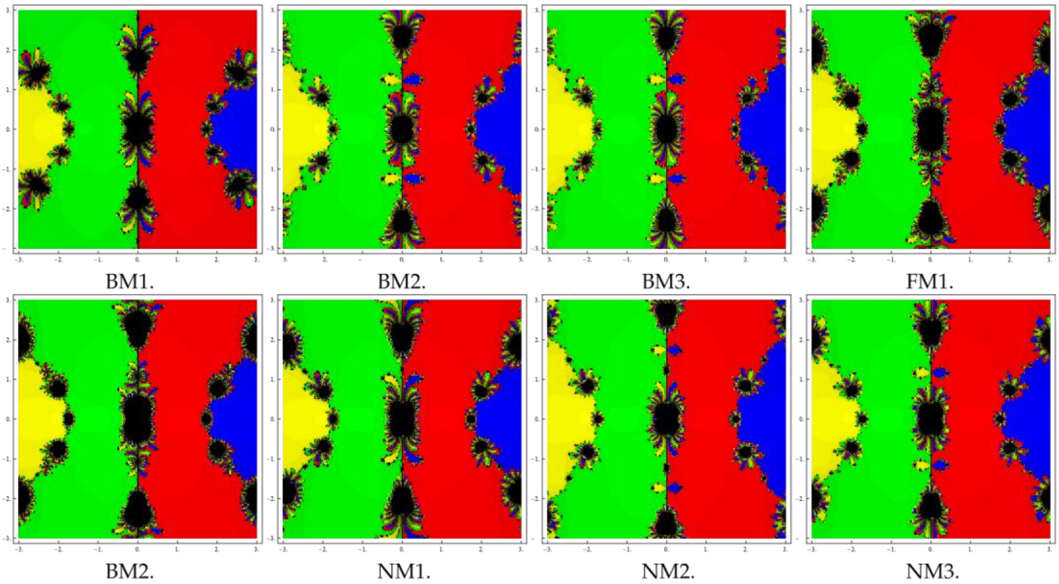

1.

Figure 3. Basins of attraction of $BM1$, $BM2$, $BM3$, $FM1$, $FM2$, $NM1$, $NM2$, $NM3$ for polynomial $P_3(z)$.

From these graphics, one can easily check the behavior and stability of any iterative procedure. If an initial guess z_0 is chosen in a region where different basins meet each other, it is difficult to guess which zero is going to be attained by the method that starts in z_0. So, the selection of z_0 in such a region is not preferable. Both the black zone (divergent zone) and the zones with different colors are not suitable to consider the initial guess z_0 when we want to acquire a unique root. The most adorable pictures can be seen along the boundaries between the basins of attraction. These boundaries have fractal-like pictures and belong to the cases where the method is more demanding with choice of initial point. At such regions, the dynamic behavior of the initial guess is more unpredictable.

6. Conclusions

In this research article, we have developed a class of optimal eighth order methods for locating multiple roots of nonlinear equations with known multiplicity. The analysis of the order of convergence has been discussed, that proves the order eighth under well-known assumptions regarding the nonlinear function whose zeros we are looking for. Some particular cases have been presented and their performance has been compared with well-known methods available in literature. The robustness of new algorithms can be judged by the fact that the accuracy in the successive approximations to the solution is much better compared to the accuracy of existing ones. Moreover, the used CPU time in execution of program is less than that of the CPU time taken by the existing techniques in majority of the cases. These conclusions have also been verified by similar numerical testing on many other different problems.

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