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A Variational Framework for the Cosmological Constant: Reconciling Classical and Quantum Descriptions

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Abstract: The manifestly-covariant Hamiltonian structure of classical General Relativity is shown to be associated with a path-integral synchronous Hamilton variational principle for the Einstein field equations. A realization of the same variational principle in both unconstrained and constrained forms is provided. As a consequence, the cosmological constant is found to be identified with a Lagrange multiplier associated with the normalization constraint for the extremal metric tensor. In particular, it is proved that the same Lagrange multiplier identifies a 4-scalar gauge function generally dependent on an invariant proper-time parameter *s*. Such a result is shown to be consistent with the prediction of the cosmological constant based on the theory of manifestly-covariant quantum gravity.

1. Introduction: Current Status of Hamiltonian Theories

A fundamental aspect of the standard formulation of General Relativity (SF-GR), i.e., the Einstein field equations [1,2], concerns its Hamiltonian representation. Its importance for establishing also the corresponding self-consistent quantization theory, namely, Quantum Gravity, is well known. However, the same representation can also be useful to pinpoint crucial aspects of SF-GR itself and display the difference among classical variational treatments of GR available in literature. This concerns in particular the comparison of past approaches proposed since the original variational theory due to Einstein himself [3] and the attempt to its Hamiltonian representation by Dirac [4], with the manifestly-covariant framework adopted here, which lays also at the basis of the recently-developed theory of covariant quantum gravity (CQG-theory, see [5,6]).

More precisely, the goal of this paper is to investigate both the mathematical and physical interpretations of the cosmological constant (CC) [7] arising in the context of the manifestly-covariant theory and to determine its possible parametric dependences. In particular, we intend to show that the CC takes the general form

$$\Lambda_{cl} \equiv \Lambda_{cl}(s), \tag{1}$$

which is consistent with its quantum prescription based on CQG-theory recently reported [8]. We intend to show that, instead, at the classical level the same CC remains undetermined, being $\Lambda_{cl}(s)$ an arbitrary, i.e., gauge, smooth real function with s being a suitably-defined proper time along the field-geodetics.

The task is achieved by introducing a novel representation for the relevant Hamiltonian variational principle which has close similarities with the analogous variational principle

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holding for discrete classical dynamical systems in Classical Mechanics. In doing this, we shall depart from the customary realization of Hamiltonian theories for SF-GR to be found in the prevailing literature [9], which ultimately dates back to Dirac [10–14]. The new theoretical framework proposed here is, instead, inspired to the reduced-dimensional manifestly-covariant Hamiltonian theory recently discovered in [15] and is based on the construction of a path-integral variational principle for the continuous Hamilton equations of GR (see Equation (12) below) and the resulting continuous Hamiltonian classical dynamical system (CDS) which are associated with the Einstein field equations. In particular, its basic feature is that of satisfying the following attributes at the same time, namely to be:

- *Manifestly covariant*, i.e., *frame-symmetric*. This means that the canonical variables, the Hamiltonian density and related functionals are necessarily set in 4-tensor form. As a consequence, the resulting Hamiltonian representation holds in arbitrary coordinate systems (i.e., GR-frames), which are mutually connected by local point transformations, i.e., diffeomorphisms of the type $r \to r^o = r^o(r)$,

  with $r \equiv \{r^\mu\}$ and $r^o \equiv \{r^{o\mu}\}$ denoting two arbitrary GR-frames.
- *Variational,* namely, as shown below (and as typical of classical Hamiltonian systems occurring in classical mechanics) the new Hamiltonian representation is prescribed via a suitable synchronous path-integral variational principle, expressed as a line-integral performed along a geodesic trajectory in terms of an invariant proper-time parameter, and for this reason referred to here as Hamilton variational principle.
- *Unconstrained,* i.e., the same Hamiltonian system can always be expressed in terms of an arbitrary independent set of canonical variables.

In addition, in the present paper we intend to show that the same Hamiltonian system satisfies also the property to be

- *Gauge-dependent,* namely, such that both the Hamiltonian density and the corresponding Lagrangian density display definite gauge properties. However, the corresponding Euler–Lagrange equations (i.e., both the Hamilton and Lagrange equations) can be shown to be generally *gauge-symmetric*, namely, independent of the gauge itself. Nevertheless, by adopting a suitably-constrained variational principle the same Euler–Lagrange equations can also be equivalently modified in a way that they become gauge-dependent, namely to include an explicit additive gauge term which depends linearly on the CC. Accordingly, the same CC acquires the connotation of a 4-scalar gauge function which at the classical level remains undetermined.

Both gauge properties indicated here, and in particular the property of the CC which motivates the present paper, are features which, as explained below, depart in several respects from previous non-manifestly covariant and non-gauge invariant variational approaches to the Einstein equations [1,2].

However, this is not the only difference. In fact, if the Hamiltonian system is not set in tensor form, so that the canonical variables and the Hamiltonian density are not 4-tensors, then the same Hamiltonian system becomes frame-dependent. An example of this type is represented by the ADM approach and the so-called Ashtekar-variable representation of the Einstein–Hilbert Lagrangian density [16–22], achieved by introducing a preliminary 3+1 foliation splitting of space-time with respect to the time-coordinate, which as such is necessarily non manifestly-covariant. We notice that in principle the 3+1 splitting should not be a problem as the choice of the coordinates remains obviously arbitrary. However, the crucial choice which is adopted in these approaches concerns the prescription of the canonical state, i.e., the Lagrangian generalized coordinates and the conjugate momenta. In fact, these variables are realized by means of non 4-tensor fields. The consequence is that the corresponding Hamiltonian theory defined in this way typically is not preserved in form under the effect of arbitrary local coordinate transformations, such as a boost, which mix the time-coordinate with space-coordinates. Therefore, besides being inherently complex and cumbersome due to the nature itself of the 3+1 foliation and the induced tensor properties on the 3D space-hypersurfaces, Dirac, ADM, and all related Hamiltonian representations become frame-dependent (i.e., dependent on the choice of coordinates). This feature is in conflict with the principle of objectivity set by the requirement that all physical laws should maintain their form in arbitrary GR-frames [23].

As such they violate, already at the classical level, the principle of manifest covariance, i.e., the requirement set by Einstein theory of GR of representing all physical laws and observables in 4-tensor form with respect to a suitable background space-time structure of the type

$$\left\{ \mathbf{Q}^4, \widehat{g} \right\}, \tag{2}$$

with $\mathbf{Q}^4$ being a 4-dimensional differential manifold and $\widehat{g} \equiv \{\widehat{g}_{\mu\nu}\}$ a background classical gravitational field metric tensor to be identified with a particular solution of the Einstein field equations. As a further critical feature, the same canonical variables are typically not independent, being generally subject to constraints between generalized Lagrangian coordinates and canonical momenta [24]. Therefore, although the variational property indicated above remains inapplicable in such a context, also a true Hamiltonian structure is effectively missing. Indeed, it is well known that even certain classes of non-Hamiltonian systems can always be reduced to suitably-constrained Hamiltonian systems (sic). An example of this type is provided by the double-Hamiltonian representation of the Schroedinger CDS [25], whereby the state of the same CDS, while being generally non-canonical, can nevertheless be represented in terms of two suitably-coupled Hamiltonian systems.

The consequences of such a type of setting are serious: (1) At the classical level the correct gauge properties of SF-GR, which usually hold in classical field theory, are now prohibited (see [26] and Section 2). (2) Standard canonical quantization methods become inapplicable. (3) Both at classical and quantum levels the so-called principle of objectivity is violated, namely, the fundamental requisite of retaining the same (tensorial) form in arbitrary coordinate systems (GR-frames) is not fulfilled any more.

1.1. Background on the Cosmological Constant

It is well-known that the cosmological constant (CC) Λ was considered for some time by Einstein as meaningless despite the fact that in his original formulation of General Relativity (GR), now referred to as standard formulation of GR (SF-GR) [1,2], its introduction was crucial to warrant the existence of stationary cosmological solutions to his namesake tensor field equation and corresponding set of tensor field components [3,7]. Accordingly, Λ was identified with a universal 4-scalar constant. Einstein observed in fact that the inclusion of CC in the field equations preserved their divergence-free conservation law. This rules out possible functional dependences of Λ on single coordinates, like coordinate-time or spatial coordinates. However, long-term experimental evidence based on astrophysical observations of the large-scale structure of the universe [27] has shown that the CC can be given a definite value, so that its inclusion in the same equations has become nowadays a well-established part of GR theory. Nevertheless, its physical origin still emerges as an unsolved issue of outmost importance for its possible conceptual implications, both in GR and in reference to quantum gravity or emergent gravity theories [28,29].

To address the problem, several theoretical approaches have been developed in the past and recent literature, in which the nature of Λ is regarded as being due to either classical or quantum contributions. In the context of modified classical gravity theories these include (1) cosmological constant in modified supergravity theories [30,31]; (2) possible torsion effects, i.e., higher-order derivatives, which in the context Einstein–Cartan gravity theory characterize the energy-momentum tensor [32]; (3) the inclusion of coordinate-time [33] or coordinate-space dependences in the CC [34,35], e.g., models based on Brans–Dicke theories [36,37]. Incidentally, these theories involve preliminary 3+1 foliations of space-time [16]. Therefore, they are not frame-independent, namely, they do not hold in arbitrary GR-frames (i.e., coordinate-systems), which are connected by the group of local point transformations. Thus, for example, the prescription of the space-time 3+1 splitting, or the choice of "ad hoc" coordinate-dependencies of the CC, are typically destroyed by arbitrary local point transformations, such as a boost, which mix up time- and space-coordinates. Regarding, instead, quantum predictions of Λ, possible candidates are numerous. A historically famous one inspired to quantum field theory [38] is the one according to which Λ might be interpreted as due to the quantum vacuum. More precisely, the conjecture is that Λ should actually be identified with the total quantum-vacuum energy density arising from selected quantum fields, belonging for example to the Standard Model. This yields for Λ an estimate which exceeds typically the experimentally-observed value of Λ by nearly 120 orders of magnitude [39]. Therefore, this route leads to highly unphysical predictions and must be rejected. Partly for this reason, several alternative models have been developed to explain the expansion/acceleration of the universe as well as the CC itself (for a review see [40]). However, again some of these approaches are not set in manifestly-covariant form [41]. These include, among others, (1) scalar- or tensor-field theories based on the introduction of either scalar quantum fields [42–46] (such as the quintessence model [47–49]) or even a combination of scalar and tensor quantum fields [50,51]. Some of these theories also predict relaxation phenomena of the CC, see, for example, in [52–55]. (2) Phenomenological models associated with dark matter and/or corresponding dark energy [56–59]. (3) Perturbative calculations performed in the framework of loop quantum gravity [60,61].

Finally, various mathematical models are available in the framework of non-commutative formulations of GR based on the use of non-commutative geometry [62–64], leading to corresponding physical interpretations of the CC. A first example of this type is provided by the implementation of a Hamiltonian description to the Wheeler–DeWitt equation and the related calculation of the CC in the non-commutative scenario characterized by existence of minimal length [65]. In this reference, use is made of a Hamiltonian operator derived from preliminary space-time foliation, whereby on space-like hypersurfaces the CC is identified with an eigenvalue of the Sturm–Liouville problem associated with the Wheeler–DeWitt equation and the computation of the zero-point energy generated by graviton fluctuations. An additional work implementing non-commutative geometry models of gravity coupled to matter is provided by the authors of [66], which proposes scenarios generating running of an effective cosmological constant together with slow-roll inflation models induced by the coupling of Higgs bosons to gravity. On the other hand, contrary to these theoretical perspectives, the authors of [67] claim that assumption of a non-commutative spacetime yields a CC in the form of an integration constant, and therefore an arbitrary parameter, which is unrelated to vacuum fluctuations.

1.2. Statement of the Problem and Goals

The problem is therefore that of couching the theory, either classical or quantum, in the proper frame-independent, i.e., manifestly-covariant form. Pursuing this line of research, the present paper deals with the physical origin of Λ in the context of a suitable tensor representation of classical SF-GR, establishing also its connection with quantum gravity. For this purpose, the manifestly-covariant classical Hamiltonian formulation of the Einstein field equations developed in [15] will be adopted. We intend to prove that within SF-GR, Λ acquires appropriate tensorial properties and a precise physical meaning related to the normalization condition of the classical background gravitational field tensor $g_b \equiv \{\widehat{g}_{\mu\nu}\}$, which is realized by the requirement $g_b^{\mu\nu} g_{b\mu\nu} = 4$. Nevertheless, in such a context, it coincides with an undetermined 4-scalar which identifies a gauge 4-scalar function. The proof is based on the gauge properties of the variational formulation for the Einstein field equations, which is realized either by the gauge transformation property of the Lagrangian density [26] or by the indeterminacy of Lagrange multipliers associated with suitable physical constraints (see discussion below). The consequence is that Λ acquires a purely quantum nature, so that its precise prescription depends on quantum theory only, and in particular on quantum gravity theory.

The scheme of the presentation is as follows. First, in Section 2, a review of the treatment of the CC in asynchronous variational principles is presented. In Section 3, the manifestly-covariant Hamiltonian structure of the Einstein field equations is recalled and its connection with the same Einstein equations is pointed out. In Section 4, a new form of the variational principle is shown to hold for the continuum Hamilton equations. Its crucial feature is that the corresponding variational functional is identified with a Hamilton functional prescribed in terms of a suitably defined path integral. Then, in Section 5, a constrained variational principle is determined which yields exactly the Einstein field equations and in which the classical CC is shown to identify a gauge function. The consistency is demonstrated between the present conclusions and the recent prediction of CC based on the theory of covariant quantum gravity reported in [8]. Final conclusions are drawn in Section 6.

2. The Cosmological Constant in Asynchronous Variational Principles

In order to characterize the manifestly-covariant variational approach developed in the present manuscript, it is instructive to preliminary summarize the most representative variational treatments of the Einstein field equations reported in the literature for SF-GR, with specific focus on the corresponding gauge invariance properties of the Lagrangian functional and the related concept of the cosmological constant term which arises in such a framework. In this regard, the two main examples of literature variational treatments are represented by the Einstein–Hilbert and the Palatini variational approaches. In both cases, the Lagrangian is identified with

$$L = L_{EH} + L_F, \tag{3}$$

where L_{EH} denotes the Einstein–Hilbert vacuum field Lagrangian

$$L_{EH} \equiv -\frac{c^3}{16\pi G} R, \tag{4}$$

with R being the Ricci 4-scalar, which is assumed to be a function of variational fields, whereas L_F is a prescribed external source field Lagrangian. The corresponding Lagrangian density L is then obtained as $\mathcal{L} = \sqrt{-g} L$, where g denotes here the determinant of the metric tensor which comes from the representation of the configuration-space volume element as $d\Omega = d^4x\sqrt{-g}$. We remark that both $\sqrt{-g}$ and L are separately not 4-tensors. Because the derivation of the Einstein equations in both Einstein–Hilbert and Palatini variational principles requires variation of the volume element $d\Omega$, so that the variational Lagrangian density L does not satisfy the property of manifest covariance, these approaches are referred to as asynchronous variational principles (see the extended discussion given in [26]). These are characterized by different choices of the functional class, to be denoted as $\{Z\}$. To illustrate the issue, we restrict ourselves and without loss of generality to the case of the vacuum

Einstein equations. In the original Einstein–Hilbert variational approach [1], $\{Z\} \equiv \{Z\}_E$ is identified with the ensemble of symmetric 4-tensors $g_{\mu\nu}(r)$ (generalized coordinates) defined as

$$\{Z\}_E = \left\{ \begin{array}{l} Z_1 \equiv g_{\mu\nu}(r) \in C^k(\mathbb{D}^4) \\ f_1(Z_1) = g^{\alpha k} g_{\beta k} - \delta^\alpha_\beta = 0 \\ \Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{(C)\alpha\beta}(g) \\ g_{\mu\nu}(r)|_{\partial\mathbb{D}^4} = g_{\mu\nu\mathbb{D}}(r) \\ w^\mu(Z_1, \partial_\mu Z_1)|_{\partial\mathbb{D}^4} = 0 \end{array} \right\}, \quad g_{\mu\nu} = g_{\nu\mu}. \tag{5}$$

Here, $k \geq 3$ is set to warrant the existence of C^1 solutions for $g_{\mu\nu}$ in $\mathbb{D}^4$ which are continuous on $\partial\mathbb{D}^4$, $\Gamma^\mu_{(C)\alpha\beta}$ denote the customary Christoffel symbols evaluated in terms of the variational field $g_{\mu\nu}$ and w^μ is the 4-vector $w^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{(C)\alpha\beta} - g^{\alpha\mu}\delta\Gamma^\beta_{(C)\alpha\beta}$, which depends both on $g_{\mu\nu}$ and its partial derivatives. The constraint $f_1(Z_1) = 0$ warrants that the variational tensor $g_{\mu\nu}(r)$ raises/lowers tensor indices and is normalized to 4. Notice that the choice of the boundary condition for w^μ involves the prescription of the partial derivative of $g_{\mu\nu}$ on the boundary $\partial\mathbb{D}^4$. An alternative possible definition of $\{Z\}_E$ that avoids such a type of boundary condition can be found in [9]. In this case, however, the variational functional $S(Z)$ needs to be modified by means of the introduction of a surface-term contribution.

The set of Euler–Lagrange equations associated with the Einstein–Hilbert variational principle in the functional class $\{Z\}_E$ can be conveniently written in symbolic representation in terms of the variational Lagrangian density L as

$$\frac{1}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} = 0, \tag{6}$$

where the partial derivative with respect to the continuum Lagrangian coordinate field $g^{\mu\nu}$ must be performed keeping constant the connections.

The second approach to be mentioned is the one referred to in the literature as the Palatini variational principle [2,9]. This is realized by considering both the metric tensor $g_{\mu\nu}$ and the connections $\Gamma^\mu_{\alpha\nu}$ as independent continuum Lagrangian coordinates. As a consequence, the functional class is identified with $\{Z\} \equiv \{Z\}_{Pal}$, represented by the ensemble of symmetric variational fields $g_{\mu\nu}(r)$ and $\Gamma^\mu_{\alpha\nu}(r)$. This is defined as

$$\{Z\}_{Pal} \equiv \left\{ \begin{array}{l} [Z_1, Z_2] \equiv \left[g_{\mu\nu}(r), \Gamma^\mu_{\alpha\nu}(r)\right] \in C^k(\mathbb{D}^4) \\ f_1(Z_1) = g^{\alpha k} g_{\beta k} - \delta^\alpha_\beta = 0 \\ g_{\mu\nu}(r)|_{\partial\mathbb{D}} = g_{\mu\nu\mathbb{D}}(r) \end{array} \right\}, \tag{7}$$

$\mu\nu$ 4 $\mu\nu$

$$\Gamma\mu\alpha\nu\,(r)|\partial D4 = \Gamma\mu\alpha\nu D\,(r)$$

with $k \geq 3$. The Euler–Lagrange equation corresponding to the functional setting $\{Z\}_{Pal}$ are obtained by noting that the variation with respect to $g^{\mu\nu}$ recovers again the symbolic Euler–Lagrange equation given by Equation (6). Instead, the extremal equation obtained by considering the variation with respect to $\Gamma_{\alpha\gamma}{}^{\beta}$ can be expressed in symbolic form as

$$\nabla_\alpha \left[\frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}} \right] = 0, \tag{8}$$

which reduces to the so-called metric-compatibility condition determining the Christoffel symbols [1].

A critical issue of the asynchronous variational formulations lies in the lack of basic gauge invariance properties which should characterize standard variational theories of continuum classical fields. The feature is ultimately related to the adoption of a non-tensorial variational Lagrangian density L and gives rise to continuum field theories which are intrinsically non-gauge invariant. It is important to stress, however, that the property of gauge invariance should be regarded as a mandatory feature of variational field theories in general. This demands that gauge invariance should be fulfilled both by variational and extremal continuum fields, the latter being identified with the solutions of the Euler–Lagrange equations determined by the variational principle. As a consequence, also the variational functional and the corresponding variational Lagrangian, together with the corresponding extremal quantities, should be necessarily determined up to a suitable gauge contribution. However, this property is violated both in the Einstein–Hilbert and Palatini asynchronous approaches.

In detail, the type of gauge-invariance property considered here is provided by the trivial gauge transformation acting on the variational field Lagrangians in terms of an arbitrary constant 4-scalar $C = const.$ by means of the transformation

$$L \to L + C. \tag{9}$$

In the standard theory of variational principles, the adding of a constant in the Lagrangian must leave invariant the extremal equations. Instead, in the asynchronous framework, it follows that the Lagrangian density L transforms necessarily as

$$\mathrm{L} \to \mathrm{L} + \sqrt{-g}C. \tag{10}$$

The introduction of the additive constant C changes in a non-trivial way the form of the Einstein field equations, generating a contribution in the extremal equations equal to $-\frac{1}{2}Cg_{\mu\nu}$. This term can effectively be regarded as a cosmological-constant term, upon letting for the cosmological constant

$\Lambda \equiv -\frac{1}{2}C$. The lack of such gauge-invariance property of action principles in curved space-time must be regarded as a relevant problem affecting the foundations of variational theory for continuum fields. Some literature works have dealt with the subject in the past. A mention deserves the approach known as Non-Gravitating Vacuum Energy Theory developed in the framework of non-Riemannian

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geometry [68–70]. Specifically, the latter is based on replacing $\sqrt{-g}$ by a non-Riemannian measure density Φ for the volume element which is not dependent of the metric and which is at the same time a total divergence, so that upon introducing the transformation (9) leads to the Lagrangian density L to transform as L → L +Φ*C*. Then, as Φ is a total divergence, this realizes a symmetry.

In the present context, however, attempts based on non-Riemannian alternative geometry formulations are excluded a priori, while validity of the standard formulation of GR with Riemannian geometry is assumed. It follows that the violation of the basic gauge invariance displayed here for the variational treatment of gravitational field equations in SF-GR appears as a serious inconsistency, and is in conflict with the gauge-invariance properties of other continuum fields, for example the electromagnetic field in Maxwell theory. The conclusion is that in the context of asynchronous variational treatments of SF-GR, due to the absence of gauge invariance properties, the cosmological constant cannot be associated with a gauge transformation or neither acquire the meaning of a gauge term of some sort, being related instead to a lack of gauge invariance property. On the other hand, because, as in the original Einstein's treatment, the cosmological-constant term in the field equations is required to satisfy the differential constraint

$$\nabla_\alpha(\Lambda \hat{g}_{\mu\mu}) = 0, \tag{11}$$

it follows that Λ is considered a universal constant, i.e., an observable of SF-GR, although its precise value remains undetermined by the classical asynchronous variational principle.

3. Manifestly-Covariant Hamiltonian Structure of SF-GR

In this section, the manifestly-covariant Hamiltonian structure of SF-GR is introduced, which realizes for the gravitational field the deDonder-Weyl manifestly-covariant canonical representation to the variational dynamics of continuum fields [71–73]. To begin with, we recall some of its peculiar characteristics. A crucial one lies in the adoption of independent and symmetric Lagrangian variables $g \equiv \{g_{\mu\nu}\} \equiv \{g_{\nu\mu}\}$ associated with the physical properties of the gravitational field. However, these are to be distinguished from the background symmetric metric tensor $g_b \equiv \{\hat{g}_{\mu\nu}\}$ which determines instead the geometric properties of the space-time and raises/lowers tensor indices [15]. Hereon, for definiteness, g_b identifies an in principle arbitrary particular solution of the Einstein field equations, with g being an in principle arbitrary and independent real symmetric tensor which generally differs from g_b. Then, the classical Hamiltonian structure of SF-GR is represented by a set $\{x_R, H_R\}$, formed by an appropriate 4-tensor canonical state $x_R(s) \equiv (g_{\mu\nu}, \pi^{\mu\nu})$, which is parametrized in terms of a suitable 4-scalar parameter s (proper-time), and a 4-scalar classical Hamiltonian density H_R. The proper-time parameter is defined along field geodesics on the background space-time with metric tensor $g_{b\mu\nu}$ by the differential identity $ds^2 = g_{b\mu\nu} dr^\mu dr^\nu$. The proper-time s therefore realizes an invariant parameter in terms of which the dynamical evolution of the Hamiltonian state is performed, and it must be distinguished from the coordinate-time used in non-manifestly covariant theories relying on 3+1 space-time decomposition. Then, by construction $x_R(s)$ fulfills a corresponding set of continuum Hamilton equations

$$\begin{cases} \frac{dg_{\mu\nu}}{ds} = \frac{\partial H_R}{\partial \pi^{\mu\nu}}, \\ \frac{d\pi^{\mu\nu}}{ds} = -\frac{\partial H_R}{\partial g_{\mu\nu}}, \end{cases} \tag{12}$$

which satisfies an initial-value condition of the type

$$x_R(s_o) \equiv (g_{\mu\nu}(s_o), \pi^{\mu\nu}(s_o)), \tag{13}$$

with $g_{\mu\nu}(s_o)$ and $\pi^{\mu\nu}(s_o)$ denoting two suitable initial tensor fields and s_o an initial proper-time. As a consequence the same equations can be viewed as canonical evolution equations for $x_R(s)$. As such, provided $H_R = H_R(x_R, s)$ is sufficiently regular, the same equations determine uniquely the proper-time evolved canonical state $x_R(s)$ in terms of an (in principle arbitrary) initial condition of the type (13). Here the notations are standard according to those in [8] (see also [5,15]). Thus, s is the proper-time along an arbitrary field geodetics $r(s) \equiv \{r^\mu(s)\}$ associated with g, which crosses the 4-position r^μ at proper-time s, while $\frac{d}{ds} = \left.\frac{d}{ds}\right|_s + \frac{d}{ds}\Big|_r$ is the covariant s-derivative operator, where

$\left.\frac{d}{ds}\right|_s \equiv t^\alpha \widehat{\nabla}_\alpha$ identifies the directional covariant derivative, with $t^\alpha = \frac{dr^\alpha(s)}{ds}$, and $\frac{d}{ds}\Big|_r$ is the covariant s-partial derivative prescribed according to the same reference (see Equation (A9) in Appendix B therein reported). As a consequence, this warrants that identically $\frac{d}{ds}\widehat{g}_{\mu\nu} = \frac{d}{ds}\widehat{g}^{\mu\nu} = 0$ even in the general case of a non-stationary background metric tensor, i.e., of the form $\widehat{g}(r, s)$ [8]. Furthermore, the Hamiltonian density is identified with the function

$$H_R(x_R) \equiv T_R + V, \tag{14}$$

where T_R and $V(g, r(s), s) \equiv V_o + V_F$ denote the effective kinetic and the normalized effective potential densities. In particular, the first one takes the form

$$T_R \equiv \frac{1}{2\alpha L}\pi_{\mu\nu}\pi^{\mu\nu}, \tag{15}$$

with α and L being suitably-prescribed dimensional constant 4-scalars identified according to the treatment given in [5]. In addition, $V_o \equiv h\alpha L\left(g^{\mu\nu}\widehat{R}_{\mu\nu} - 2\Lambda\right)$ and V_F identify respectively the vacuum and external potential contributions, the first one carrying also the cosmological constant term, i.e., linearly proportional to Λ. Notice that here h is the variational weight-factor $h = 2 - \frac{1}{4}g^{\alpha\beta}g^{\mu\nu}\widehat{g}_{\alpha\mu}\widehat{g}_{\beta\nu}$ while all hatted quantities are evaluated in terms of the background field tensor $\widehat{g}$, so that, in particular, $\widehat{R}_{\mu\nu}$ is the corresponding Ricci tensor function $\widehat{R}_{\mu\nu} \equiv R_{\mu\nu}(\widehat{g})$. Thus, in terms of these definitions, one obtains

$$\begin{cases} \frac{dg_{\mu\nu}}{ds} = \frac{\pi_{\mu\nu}}{\alpha L}, \\ \frac{d\pi_{\mu\nu}}{ds} = -\frac{\partial V}{\partial g^{\mu\nu}}, \end{cases} \tag{16}$$

where the first equations determines the canonical momentum $\pi_{\mu\nu}$ in terms of the "generalized velocity" $\frac{dg^{\mu\nu}}{ds}$, while the second one provides a dynamical equation for the remaining variables. The connection with the Einstein field equations follows in straightforward way. This is obtained in particular by requiring the initial condition

$$x_R(s_o) = \left(g_{\mu\nu}(s_o) \equiv \widehat{g}_{\mu\nu}(s_o), \pi^{\mu\nu}(s_o) \equiv \widehat{\pi}^{\mu\nu}(s_o) = 0\right), \tag{17}$$

where $\hat{g}_{\mu\nu}(s_o)$ is the background metric tensor evaluated at $(r(s_o), s_o)$ and, consistent with the requirement (2), is necessarily a solution of the initial stationary equation

$$\left.\frac{\partial V(g,r(s),s)}{\partial g^{\mu\nu}}\right|_{g=\hat{g},s=s_o} = 0, \tag{18}$$

while $\hat{\pi}^{\mu\nu}(s_o) = 0$ is the corresponding initial null canonical momentum. Then, one can prove that for all $s \geq s_o$, $x_R(s) = g_{\mu\nu}(s) \equiv \hat{g}_{\mu\nu}(s), \pi^{\mu\nu}(s) \equiv \hat{\pi}^{\mu\nu}(s) \equiv 0)$, with $\hat{g}_{\mu\nu}(s)$ being for all $(r(s), s)$ solution of the stationary equation:

$$\left.\frac{\partial V(g,r(s),s)}{\partial g_{\mu\nu}}\right|_{g=\hat{g}} = 0. \tag{19}$$

Indeed, the initial conditions (17) together with Equation (18) imply that $\left.\frac{d\pi^{\mu\nu}}{ds}\right|_{s=s_o} = 0$. As a consequence, it follows that identically $\hat{\pi}^{\mu\nu}(s) \equiv 0$ so that $\frac{dg^{\mu\nu}}{ds} \equiv 0$ with $g_{\mu\nu}(s) \equiv \hat{g}_{\mu\nu}(s)$ solution of Equation (19). It is important to stress here the meaning of Equations (18) and (19). In fact, straightforward algebra shows (see also [8]) that they coincide in both cases with the Einstein field equations in the presence of sources, namely,

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu} + \Lambda\hat{g}_{\mu\nu} = \frac{8\pi G}{c^4}\hat{T}_{\mu\nu}, \tag{20}$$

i.e., when they are evaluated respectively at $(r(s_o), s_o)$ and $(r(s), s)$, and in which again all hatted quantities are functions of $\hat{g}_{\mu\nu}$.

4. New Hamiltonian Representation: Constraint-Free Hamilton Variational Principle

Let us now prove that the continuum Hamilton Equation (12) can actually be associated with a novel Hamiltonian representation of GR. The result is based on a new variational formulation which departs from the one effectively realized in [15,26]. For this purpose, we wish to show that the same equations admit an equivalent constraint-free synchronous variational formulation in standard form, i.e., in which the variational functional is prescribed in terms of a path integral rather than a configuration-space integral as in the references indicated above. For its close similarity with the standard theory of Hamiltonian dynamical systems, this will be referred to as (modified) *Hamilton variational principle*. Let us introduce for this purpose the Hamilton functional

$$J(x_R) = \int_{s_o}^{s_1} ds \left(\pi^{\mu\nu}(s)\frac{dg_{\mu\nu}(s)}{ds} - H_R(x_R(s),s)\right), \tag{21}$$

and the synchronous variational principle

$$\delta J(x_R) = 0, \tag{22}$$

with

$$L_R\left(g(s),\frac{dg}{ds},s\right) = \pi^{\mu\nu}(s)\frac{dg_{\mu\nu}(s)}{ds} - H_R(x_R(s),s), \tag{23}$$

denoting the Legendre-conjugate Lagrangian density. Here, the variation operator δ is prescribed in terms of the Lagrangian coordinates $g_{\mu\nu}(s)$ and conjugate momenta $\pi^{\mu\nu}(s)$, i.e., of the corresponding canonical state x_R and denoted by the symbol δx_R. Therefore, the action of the variation operator δ on $J(x_R)$ is identified with the synchronous Frechet derivative

$$\delta J(x_R) \equiv \frac{d}{d\xi}\Psi(\xi)\Big|_{\xi=0} = 0, \quad (24)$$

being $\Psi(\xi)$ the smooth real function defined as $\Psi(\xi) = J(x_R + \xi\delta x_R)$ and $\xi \in]-1, 1[$ to be considered here an independent variable. Thus, in particular, this means that the boundary conditions $x_R(s_o)$ and $x_R(s_1)$ occurring at the boundary proper-times s_o and s_1 are set. Namely, they are of the type

$$x_R(s_i) = g_{\mu\nu}(s_i), \pi^{\mu\nu}(s_i)), \quad (25)$$

in which, for $i = 0, 1$, the boundary tensor fields $g_{\mu\nu}(s_o)$ and $\pi^{\mu\nu}(s_o)$ remain in principle arbitrary. Notice here that $x_R(s_1)$ cannot be independent of $x_R(s_o)$ and actually is assumed to be suitably prescribed. Furthermore, the following is understood.

(a) δx_R identifies the synchronous variation

$$\delta x_R = x_R(s) - x_{R1}(s), \quad (26)$$

with $x_R(s)$ and $x_{R1}(s)$ being two different canonical states. Notice that in Equation (22) the tensor components of the variation δx_R are considered independent (for this reason the same variational principle can be referred to as constraint-free).

(b) The path integral in the functional $J(x_R)$ is performed along a generic finite-length field geodetics $r(s)$ so that $x_R(s)$ is parametrized in terms of it, namely, letting $x_R(s) \equiv x_R(r(s), s)$. Therefore, the Hamiltonian density and the background metric tensor are analogously parametrized and hence are of the form $H_R \equiv H_R(x_R(s), r(s), s)$ and $g_b(s) \equiv g_b(r(s), s)$.

(c) The variation operator δ in the variational principle (24) is synchronous, i.e., it is such that it leaves invariant both the proper time s and the field geodetics $r(s)$ so that identically

$$\begin{cases} \delta(ds) \equiv 0 \\ \delta r(s) = 0 \end{cases}, \quad (27)$$

and furthermore it leaves similarly invariant also the boundary conditions (25), so that

$$\delta(x_R(so)) = \delta(x_R(s_1)) = 0. \quad (28)$$

(d) The variation operator δ leaves invariant also the background field tensor $g_b(s) \equiv g_b(r(s), s)$ which is considered prescribed and such that

$$\delta g_b(s) \equiv 0. \quad (29)$$

This type of requirement, although unprecedented in constrained dynamics [24], is not a constraint at all. In fact it actually leaves unaffected the synchronous variation δx_R and can be regarded as part of the definition of the synchronous operator δ itself.

Then, in view of the previous prescriptions, the same variational principle delivers Euler–Lagrange equations in Hamiltonian form which coincide with the canonical equations (12). In fact, the functional derivatives of $J(x_R)$ yield explicitly

$$0, \qquad \left(\frac{\delta J(x_R)}{\delta \pi^{\mu\nu}} = \frac{dg_{\mu\nu}}{ds} - \frac{\partial T_R}{\partial \pi^{\mu\nu}} = \right.$$

$$\delta\delta\mathcal{J}(x\mu\nu R) = \frac{V}{g_{\mu\nu}} = -d\pi ds\mu\nu - \partial\partial 0, \qquad (30)\ g$$

with the solutions being subject to the boundary conditions (25). Therefore, this shows that just as in analytical mechanics [74], Hamilton equations for the SF-GR can be equivalently determined in terms of a Hamilton variational principle in which the variational functional is a path-integral of the form (21). This identifies a unique feature of the manifestly-covariant synchronous approach, which allows to cast the variational functional as a line integral, while it remains excluded from non manifestly-covariant asynchronous treatments. In addition, by construction the same boundary conditions (25) are assumed mutually consistent so that Equations (25)–(30) admit a unique solution which holds for arbitrary $s_o < s_1$ and $s \in (s_o, s_1)$. The basic implication is therefore that the same boundary-value problem actually determines a Hamiltonian CDS which is represented by the bijection

$$x_R(so) \equiv g_{\mu\nu}(s_o), \pi^{\mu\nu}(s_o)) \Leftrightarrow x_R(s) \equiv g_{\mu\nu}(s), \pi^{\mu\nu}(s)). \qquad (31)$$

In terms of the Lagrangian density (23), the corresponding Lagrange equations (and related Lagrangian variational principle) follow at once from Equations (30) and are given by the manifestlycovariant tensor equations

$$\frac{d}{ds}\frac{\partial LR}{\partial\left(\frac{dg_{\mu\nu}}{ds}\right)} - \frac{\partial LR}{\partial g_{\mu\nu}} = 0. \qquad (32)$$

Finally, it must be stressed that both the Hamiltonian and Lagrangian densities $H_R(x_R(s), s)$ and $L_R\left(g(s), \frac{dg(s)}{ds}, s\right)$ are intrinsically non-unique, being necessarily determined up to an additive gauge function of the form

$$\frac{d}{ds}F(g(s), r(s), s), \qquad (33)$$

with $F(g(s), s)$ denoting a real arbitrary 4-scalar field of class $C^{(2)}$. Nevertheless, the Hamilton Equation (30) and corresponding Lagrange Equation (32) are both unique and as such can be considered classical observables. In fact, it is then immediate to prove that indeed the function (33) is a gauge term in a proper sense, i.e., that it does not contribute neither to the Hamilton or Lagrange equations.

5. Constrained Hamilton Variational Principle

Let us now pose the problem of the construction of a variational principle directly for the Einstein field equations themselves, to be set again in the form (19) given above. More precisely, the target here is to determine a variational principle which yields Hamilton equations which coincide identically with the Einstein equations, without passing through the general representation (16) and the subsequent imposition of the particular initial conditions (17). As proved below, this can be achieved in terms of a suitably-constrained synchronous variational principle. This route is instrumental in order to display the nature of the CC as a gauge term in the manifestly-covariant variational treatment of SF-GR.

One first notices that a possible realization of such a variational principle can simply be achieved by properly setting the initial and final boundary conditions (25) in the relevant variational functional (see Equation (21)), i.e., by requiring that for $i = 0, 1$, they coincide with

$$x_R(s_i) = g_{\mu\nu}(s_i) \equiv \widehat{g}_{\mu\nu}(s_i), \pi^{\mu\nu}(s_i) \equiv \widehat{\pi}^{\mu\nu}(s_i) = 0). \tag{34}$$

However, again in a proper sense also this requirement cannot be considered as a constraint condition since, as explained below, it does not affect the class of variations of the same functional.

Nevertheless, an equivalent realization can also be achieved by making use of a constrained variational principle in a proper sense. Thus, besides setting the boundary conditions (34) the solution can be sought in the framework of the same synchronous path-integral functional indicated above, the path-integral being performed again along a generic finite-length field geodetic $r(s)$. We adopt for this purpose the standard method of Lagrange multipliers, thus introducing the functional

$$J_L(x_R) = J(x_R) + J_1(x_R) + J_2(x_R), \tag{35}$$

where, respectively, $J(x_R)$ is defined by Equation (21), while

$$J_1(x_R) = -\frac{1}{2}\int_{s_o}^{s_1} ds\,\lambda_1 \left(g^{\mu\nu} g_{\mu\nu} - 4\right), \tag{36}$$

$$J_2(x_R) = -\frac{1}{2}\int_{s_o}^{s_1} ds\,\lambda_2 \pi^{\mu\nu}\pi_{\mu\nu}. \tag{37}$$

Then, let us consider the constrained synchronous variational principle $\delta J_L(x_R) = 0$ performed in terms of independent variations of the Lagrangian coordinates, the conjugate momenta and of the two Lagrange multipliers λ_1 and λ_2, while letting again $\delta ds = 0$ and $\delta g_b = 0$. The corresponding Euler–Lagrange equations are given respectively by the equations

$$\begin{cases} \frac{\delta J_L(x_R)}{\delta \pi^{\mu\nu}} = \frac{dg_{\mu\nu}}{ds} - \frac{\partial H_R}{\partial \pi^{\mu\nu}} - \lambda_2 \pi_{\mu\nu} = 0, \\ \frac{\delta J_L(x_R)}{\delta g_{\mu\nu}} = -\frac{d\pi_{\mu\nu}}{ds} - \frac{\partial H_R}{\partial g_{\mu\nu}} - \lambda_1 g_{\mu\nu} = 0, \\ \frac{\delta J_L(x_R)}{\delta \lambda_1} = g^{\mu\nu} g_{\mu\nu} - 4 = 0, \\ \frac{\delta J_L(x_R)}{\delta \lambda_2} = \pi^{\mu\nu}\pi_{\mu\nu} = 0. \end{cases} \tag{38}$$

Then, upon identifying $g \equiv g_b$ and setting the boundary conditions (34), it follows that the previous equations reduce to

$$\begin{cases} \frac{d\widehat{g}_{\mu\nu}}{ds} = \frac{\widehat{\pi}_{\mu\nu}}{\alpha L} = 0, \\ \left.\frac{\partial V}{\partial g_{\mu\nu}}\right|_{g=\widehat{g}} + \lambda_1 \widehat{g}_{\mu\nu} = 0, \\ \widehat{g}^{\mu\nu}\widehat{g}_{\mu\nu} = 4, \\ \pi_b^{\mu\nu} = \pi_{b\mu\nu} = 0, \end{cases} \tag{39}$$

where the Lagrange multiplier λ_1 remains arbitrary. Thanks to the constraint equation associated with the undetermined Lagrange multiplier λ_2 (i.e., the fourth equation in the previous system),

$\frac{dg^{\mu\nu}}{}$ the condition of vanishing derivative $\frac{}{ds}$b = 0 provided by the first equation remains identically satisfied for all s. Furthermore, the second equation in particular delivers that a modified Einstein field equation actually holds, in which the contribution of the CC Λ is replaced with the *rescaled CC*

$$\Lambda_1 \equiv \Lambda + \lambda_1. \tag{40}$$

We notice that in the present context λ_1 can be regarded as a classical parameter, which in general can be considered an arbitrary function of the form $\lambda_1 = \lambda_1(g_b, r(s), s)$, consistent with the synchronous principle. Notice, however, that possible dependences of CC on $r(s)$ can be ruled out based on the symmetry property of the Einstein field equations. Similarly, the possible dependence on gb can occur only through 4-scalar saturations of the Ricci or Riemann tensors. The resulting equations in such cases however would depart from the standard Einstein field equations, and therefore can be considered as outside the scope of SF-GR. The conclusion is that within the same context, necessarily at most the rescaled CC is of the form (1), namely,

$$\Lambda_{cl}(s) \equiv \Lambda_1(s). \tag{41}$$

This is the main result of the paper, together with the introduction of the path-integral synchronous Hamilton variational principle (24) and the corresponding constrained principle holding for the Einstein field equations. The conclusions are therefore as follows.

(1) The Einstein field equations are actually characterized by a rescaled CC of the general type (41).

(2) The physical meaning of the rescaled CC Λ_1 is that of a gauge function, namely, an arbitrary

4-scalar Lagrange multiplier associated with the normalization constraint $g_b^{\mu\nu} g_{b\mu\nu} = 4$.

(3) Given the arbitrariness of the constant contribution Λ in Equation (40), this can always be set equal to zero. In fact, even starting with a vanishing CC Λ, the constrained synchronous variational principle warrants that a non-vanishing rescaled CC Λ_1 must occur in the Euler–Lagrange extremal equations.

(4) All the variational principles pointed out here recover the correct gauge properties pointed out in [26], whereby the Hamilton density function H_R can be replaced with $H_R + C$ or $H_R + \frac{dF}{ds}$, with C and F denoting respectively an arbitrary real constant and a differentiable function $F(g_b, r(s), s)$.

The remarkable consequence is that the rescaled CC $\Lambda_1(s)$ exhibits a functional dependence which is exactly of the type predicted by the theory of manifestly-covariant quantum gravity (CQG-theory), namely the canonical-quantization theory of the gravitational field based of the classical Hamiltonian structure $\{x_R, H_R\}$ recently developed in [5,6,15,26]. In particular, as shown in [8], in such a context the quantum contribution is found to be represented by a generally non-stationary function of the form $\Lambda CQG(s) =$ ____ __$(a\bar{h} L2)2\ r1th4\, f(s)$, (42)

with $\Lambda_{CQG}(s)$ identifying the *CGQ-cosmological constant*. Here, in addition to the notation indicated above, $\bar{h}$ is the reduced Planck constant and $f(s)$ is a strictly positive 4-scalar function constructed such that at the initial proper-time s_o (which can be set equal to zero) $f(s_o) = 1$, while in the limit $s \to +\infty$, it tends to a non-vanishing positive constant. Finally, $r_{th}{}^4$ is a suitable dimensionless 4-scalar parameter estimated in [5], which enters the prescription of the quantum probability density associated with the quantum state [75]. According to the authors of [8], from the physical standpoint $\Lambda_{CQG}(s)$ was shown to be ascribed to the nonlinear Bohm quantum vacuum interaction of the gravitational field with itself, namely produced by the self-interaction of massive gravitons. This feature explains the dependence of $\Lambda_{CQG}(s)$ in Equation (42) in terms of the squared reduced Planck constant $\hbar^2$, which is the same type of dependence carried by the Bohm potential and ultimately due to the structure of the quantum-wave equation [6]. The appearance of $\hbar^2$ therefore characterizes the solution for $\Lambda_{CQG}(s)$ as an intrinsically-quantum term of second order in $\hbar$, which retains the information of the nonlinear quantum self-interaction of massive gravitons in vacuum. Furthermore, thanks to the realization of the quantum probability density, the explicit s-dependence of $\Lambda_{CQG}(s)$ arises because of the gradients of the vacuum quantum gravitational energy density. The implication is therefore that $\Lambda_1(s)$ can actually be identified with $\Lambda_{CQG}(s)$, acquiring therefore a well-defined quantum prescription. Finally, it must be stressed that the same conclusions apply also in the more general case in which the CGQ-cosmological constant is subject to the additional quantum-driven screening mechanism pointed out in [76], which can affect the absolute magnitude of the CC but not its functional dependence established by Equation (42).

6. Conclusions

The majority of previous mainstream literature approaches to the cosmological constant are typically based on explicitly non-manifestly covariant Hamiltonian treatments of the Einstein field equations. This usually involves counter-intuitive and possibly cumbersome non-tensor representations of the canonical variables and the adoption of a non-gauge asynchronous Einstein–Hilbert variational functional. In the present paper, a new Hamiltonian representation of SF-GR has been developed which—as is customary in the theory of classical Hamiltonian systems—is based on a path-integral representation of the Hamilton variational principle. Its crucial feature is that of being manifestly-covariant in form, i.e., expressed in terms of 4-tensor representation of all fields and variables, including in particular the canonical variables, the Hamiltonian density and of course the Hamilton variational functional. As a consequence the theory becomes frame-symmetric, its form being independent of the choice of the GR-frame. The feature is of paramount importance because it permits to recover at the same time also the gauge-symmetry properties characteristic of standard Hamiltonian systems. In particular, this makes possible the adoption of a synchronous variational principle, identified here with the path-integral Hamilton functional, and the explicit inclusion of gauge fields in the Hamiltonian/Lagrangian densities as in classical field theory. Therefore, we have proved that in this way a representation of the Einstein field equations can be achieved via the Hamilton variational principle, both in unconstrained and constrained forms.

The fundamental implication reached in this paper lies in the identification of the cosmological constant in terms of a gauge scalar field, i.e., an undetermined Lagrange multiplier. In particular, this shows that generally the cosmological constant is a 4-scalar function of a suitably-defined proper time s (associated with an arbitrary local geodetics of the background field metric tensor $g\mathrm{b}_{\mu\nu}$). Remarkably the result is in agreement with the prediction of the cosmological constant obtained in the framework of manifestly-covariant quantum gravity theory.

Such conclusions are promising from the theoretical standpoint. In fact, they are based on a novel path-integral variational representation of the Einstein field equations. Its basic feature is that of being, just like the same Einstein equations, manifestly-covariant in character. This permits to display in a perspicuous and intuitive way the Hamiltonian character of GR, unveiling at the same time also its basic properties with particular reference to the role of the cosmological constant. As such the present theory can represent a useful basis and a possible new pathway for the formulation of classical GR itself and the corresponding quantum theory.

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