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Fractional Dynamics in Non-Instantaneous Impulsive Systems

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Abstract: Based on some previous works, an equivalent equations is obtained for the differential equations of fractional-order$q \in (1, 2)$ with non-instantaneous impulses, which shows that there exists the general solution forthisimpulsivefractional-ordersystems.Next,anexample is used to illustrate the conclusion.

Keywords: fractional differential equations, impulses, non-instantaneous impulses, general solution **PACS:** 02.30.Hq, 02.30.Ik

1 Introduction

Fractional differential equations has gained much attention in literature because of its applications for description of hereditary properties in many fields, and some progresses were gotten in computation methods, controllability, existence etc. for fractional differential equations [1–6]. Moreover, impulsive fractional (partial) differential equations were widely studied [7–29] due to importance in description of some processes in which sudden, discontinuousjumpsoccur,andgeneralsolutionhasbeen

$$\begin{cases} {}^{C}D_{0+}^{q}x(t) = f(t, x(t)), & (1.1a) \\ \quad q \in (1, 2),\ t \in (s_k, t_{k+1}],\ k = 0, 1, \dots, N, & \\ x(t) = g_k(t, x(t)),\ t \in (t_k, s_k],\ k = 1, 2, \dots, N, & (1.1b) \\ x(0) = x_0,\ x'(0) = \bar{x}_0,\quad x_0, \bar{x}_0 \in \mathrm{R}. & (1.1c) \end{cases}$$

here${}^{C}D_{0+}^{q}$ is the Caputo fractional derivative of order q. $f : [0, T] \times \mathrm{R} \to \mathrm{R}$ and $g_k : (t_k, s_k] \times \mathrm{R} \to \mathrm{R}$ are some appropriate functions, and g_k denote non-instantaneous impulses, and $g'_k(s_k, x(s_k))$exist (here $k = 1, 2, \dots, N$), and$0 = t_0 = s_0 < t_1 \le s_1 \le t_2 \le \dots \le t_N \le s_N \le t_{N+1} = T$.

Next, let us introduce the concept of the fractional derivative and some conclusions in Section 2, and provide main result in section 3, and give an example to show the usefulness of the obtained result.

2 Preliminaries

Definition 2.1 [39]. The left-sided Riemann-Liouville fractional integral $I_{a+}^{p}x$of order $p (p > 0)$ for function$x$ is defined as

$$I_{a+}^{p}x(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{x(\tau)}{(t - \tau)^{1-p}}\, d\tau \quad (t > a,\ p > 0),$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition2.2[39].TheCaputofractionalderivative ${}^{C}D_{a+}^{q}x$ of order $q (q > 0)$ for a function $x$can be written as

$${}^{C}D_{a+}^{q}x(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{x^{(n)}(\tau)}{(t-\tau)^{q+1-n}}\, d\tau = (I_{a+}^{n-q}D^{n}x)(t),\ t > a,\ \text{where} D = d/dt$$

and $q \in (n - 1, n)$.

Lemma 2.3 [39, 40]. If the function $g(t, x)$ is continuous, then the initial value problem

$$\begin{cases} {}^{C}D_{a+}^{q}x(t) = g(t, x(t)), & q \in (n-1, n),\ n \ge 1 \\ x^{(k)}(a) = x_a^k, & k = 0, 1, 2, \dots, n-1. \end{cases},$$

is equivalent to the following nonlinear Volterra integral equation of the second kind,

$$x(t) = \sum_{k=0}^{n-1} \frac{x_a^k}{k!}(t-a)^k + \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} g(\tau, x(\tau))\, d\tau.$$

Lemma 2.4 [31]. Let ξ and ζbe two constants. The impulsive system

$$\begin{cases} {}^{C}D_{0+}^{q}x(t) = f(t, x(t)), \\ \quad q \in (1, 2),\ t \in J = [0, T],\ t = t_k (k = 1, \dots, m), \\ \Delta x|_{t=t_k} = I_k\left(x(t_k^-)\right), \quad k = 1, 2, \dots, m, \end{cases}$$

$$\Delta x'\big|_{t=t_k} = \bar{I}_k\left(x(t_k^-)\right), \quad k = 1, 2, \ldots, m,$$

$$x(0) = x_0, \quad x'(0) = \bar{x}_0.$$

(2.1)

is equivalent to the integral equation

$$x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f ds \text{ for } t \in J_0,$$

$$x_0 + \bar{x}_0 t + \sum_{i=1}^{k} I_i(x(t_i^-)) + \sum_{i=1}^{k} (t - t_i)\bar{I}_i(x(t_i^-)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f ds$$

$$x(t) = \Big\{ + \sum_{i=1}^{k} [\xi I_i(x(t_i^-)) + \bar{\zeta} I_i(x(t_i^-))]$$

$$\left\{ \frac{1}{\Gamma(q)} \left[\int_0^{t_i} (t_i - s)^{q-1} f ds + \int_{t_i}^t (t-s)^{q-1} f ds - \int_0^t (t-s)^{q-1} f ds \right] \right.$$

$$\left. + \frac{(t - t_i)}{\Gamma(q-1)} \int_0^{t_i} (t_i - s)^{q-2} f ds \right\} \text{ for } t \in J_k,\ 1 \le k \le m. \quad (2.2)$$

provided that the integral in (2.2) exists. Here $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$ $(k = 1, 2, \ldots, m)$.

3 Main result

For convenience, let $f = f(\tau, x(\tau))$ in this section. Consider condition (1.1a) in system (1.1) by using two different approaches:

(i) $\{ {}_{C}D^{0+}{}_{q}\text{for}^{x}(tt) =\in (f_S({}_kt,,{}_tx_k({}_{+1}{}^{t}))].,$ $=$ $\{$

${}^{C}D^{sq}\text{for}_{k+}{}^{x}({}_{t}{}^{t}) =\in ({}_{S}f_k(,{}^{t},{}_{t}{}^{k}x_{+1}({}^{t}))].,$

$$\Leftrightarrow \begin{cases} x(t) = x(s_k) + x'(s_k)(t - s_k) + \frac{1}{\Gamma(q)} \int_{s_k}^t (t-\tau)^{q-1} f d\tau, \\ \text{for } t \in (s_k, t_{k+1}]. \end{cases}$$

(3.1)

(ii) $\{ {}^{C}D_{0+}\text{for}^{q}\, x(tt) =\in (sf({}_kt,,tx_{k+1}(t))] ,\subset (0, T],$

$$\Leftrightarrow \Big\{ \text{for } t \quad (s_k, t_k \quad , \quad x(t) = C_k + B_k t + x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau, \in {}_{+1}] \subset (0, T]$$

here C_k and B_k areconstants.

(3.2) Next, substituting (*i*) into system (1.1), we get

$$\begin{cases} x(t) = x(s_k) + x'(s_k)(t - s_k) + \frac{1}{\Gamma(q)} \int_{s_k}^t (t-\tau)^{q-1} f d\tau, \\ \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, \ldots, N, \\ x(t) = g_k(t, x(t)), \text{ for } t \in (t_k, s_k],\ k = 1, 2, \ldots, N, \\ x(0) = x_0,\ x'(0) = \bar{x}_0,\ x_0, \bar{x}_0 \in \mathbb{R}. \end{cases}$$

That is,

$$\tilde{x}(t) = \begin{cases} x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau \text{ for } t \in (0, t_1], \\ g_k(t, x(t)) \quad \text{for } t \in (t_k, s_k],\ k = 1, 2, \ldots, N, \\ g_k(s_k, x(s_k)) + g'_k(s_k, x(s_k))(t - s_k) + \frac{1}{\Gamma(q)} \int_{s_k}^t (t-\tau)^{q-1} f d\tau \quad \text{for } t \in (s_k, t_{k+1}],\ k = 1, \ldots, N. \end{cases}$$

(3.3)

In fact, $\tilde{x}(t)$ satisfies conditions (1.1a)–(1.1c) in system (1.1). But, we will show that $\tilde{x}(t)$ isn't a solution of system (1.1). For system (1.1), we have

$$\{\text{system}(1.1)\} \xrightarrow[\text{for all } k \in \{1,2,\ldots,N\}]{g_k(t,x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau}$$

$$\rightarrow \begin{cases} {}^{C}D_{0+}^{q}\, x(t) = f(t, x(t)), \quad t \in (s_k, t_{k+1}],\ k = 0, 1, \ldots, N, \\ x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau, \quad t \in (t_k, s_k],\ k = 1, 2, \ldots, N, \\ x(0) = x_0,\ x'(0) = \bar{x}_0,\ x_0, \bar{x}_0 \in \mathbb{R}. \end{cases} \quad (3.4)$$

$$\Leftrightarrow \begin{cases} {}^{C}D_{0+}^{q}\, x(t) = f(t, x(t)), \quad t \in (0, T], \\ x(0) = x_0,\ x'(0) = \bar{x}_0,\ x_0, \bar{x}_0 \in \mathbb{R}. \end{cases}$$

And system (3.4) is equivalent to

$$x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau \text{ for } t \in (0, T]. \quad (3.5)$$

Moreover,letting $g_k(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)_{q-1} f d\tau$ (for all $k \in \{1, 2, \ldots, N\}$) in (3.3), we get

$$x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)_{q-1} f d\tau \text{ for } t \in (0, t_1],$$

$$x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \Big| \; f d\tau \text{ for } t \in (t_k, s_k],$$

$$(t) = \Big\{ \quad k = 1, 2, \ldots, N \quad ,$$

$$x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \Big[\int_0^{s_k} (\quad \tilde{x} s_k - \tau)_{q-1} f d\tau$$

$$k \qquad f d\tau \Big] \quad \Gamma \; q$$

$$+ \frac{(t-s_k)}{(\;-1)} \int_0^{s_k} (s_k - \tau)$$

$$+ \int_s^t (t-\tau)^{q-1} \quad {}_{q-2} f d\tau \; \Big| \Big| \Big(\text{for } t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, N.$$

(3.6) Therefore, if $\tilde{x}(t)$ is a solution of system (1.1), then (3.6) is equivalent to (3.5). Thus,

$$\frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau$$

$$= \frac{1}{\Gamma(q)} \Big[\int_0^{s^k} k \quad \int_{s_k}^{t} (s-\tau)^{q-1} f d\tau + \quad (t - \tau)^{q-1} f d\tau \Big]$$

$$+ \frac{(t-s_k)}{\Gamma(\;-1)} \int_0^{s_k} (s_k - \tau) \quad q \quad {}_{q-2} f d\tau$$

for $t \in (s_k, t_{k+1}]$, $k = 1, 2, \ldots, N$. (3.7)

Eq. (3.7) is an unfit equation, which means that $\tilde{x}(t)$ isn't a solution of system (1.1). Therefore, we will regard $\tilde{x}(t)$ as an**approximatesolution**toseektheexactsolutionofsystem (1.1).

Substituting (*ii*) into system (1.1), we obtain

$$x(t) = C_k + B_k t + x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau,$$

for $t \in (s_k, t_{k+1}]$, $k = 0, 1, \ldots, N$, $\Big| \; x(t) = g_k(t, x(t))$, *for* $t \in (t_k, s_k]$, $k = 1, 2, \ldots, N$, $\Big| \Big(\Big| \; x(0) = x_0,\ x'(0) = \bar{x}_0,\ x_0, \bar{x}_0 \in$ R.

(3.8)

By initial conditions $x(s_k) = g_k(s_k, x(s_k))$ and $x'(s_k) = g'_k(s_k, x(s_k))$ (here $k = 1, 2, \ldots, N$), we obtain $B_0 = 0$, $C_0 = 0$,

$$B_k = g'_k(s_k, x(s_k)) - \bar{x}_0 - \cancel{\Gamma(q1-1)} \int_0^{s_k} (s_k - \tau)_{q-2} f d\tau,$$

$k = 1, 2, \ldots, N$. (3.9) and

$$C_k = g_k(s_k, x(s_k)) - x_0 - \Gamma(\underline{1}q) \int_0^{s_k} (s_k - \tau)_{q-1} f d\tau$$

$$- g'_k(s_k, x(s_k)) s_k + \frac{s_k}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau,$$

$k = 1, 2, \ldots, N$, (3.10) Substituting (3.9)-(3.10) into (3.8), we get

$$x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau \text{ for } t \in (0, t_1],$$

$$(t) = \Big\{ \quad (\;,\; (\;)) - \frac{}{\Gamma \; q} \qquad f d\tau$$

$$\frac{1}{} \quad t \quad {}^{q-1} f d\tau$$

$$+(t - s_k) \; g'_k(s_k, x(s_k)) - \frac{1}{} \int^{s_k} (s_k - \tau)^{q-2} f d\tau$$

$$g_k(t, x(t)) \text{ for } t \in (t_k, s_k],\ k = 1, 2, \ldots, N,$$

$$g_k \; s_k \; x \; s_k \quad (1) \int_{0}^{s_k} (s_k - \tau)_{q-1} x \quad \int$$

$$+ \Gamma(q) \quad 0 \, (t - \tau)$$

$$\Big[\qquad \Gamma(q-1) \quad 0 \qquad \Big]$$

$$\text{for } t \in (s_k, t_{k+1}], \ldots, N \quad k = 1, 2 \quad .$$

(3.11) In fact, Eq. (3.11) satisfies conditions (1.1a)–(1.1c) and

$$\{\text{Eq. } (3.11)\} \quad g_k{}_{q-1} f d\tau \; (t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau) \xrightarrow{} \text{ for all } k \in \{1, 2, \ldots, N\}$$

$$\Gamma(q) \quad 0 \quad \Big(\Big| \; x_0 + \bar{x}_0 t + \frac{1}{}$$

$$(t) = \Big\{ \text{ for } t \quad , \quad x_0 + \bar{x}_0 t + \frac{}{\Gamma(q)} \int_0 (t-\tau)^q \quad \int^t (t - \tau)^{q-1} f d\tau, \in (0, t_1]$$

$$\text{for } t \in (t_k, s_k],\ k = 1, 2$$

$$1 \quad t \quad {}_{-1} f d\tau,$$

$$\to x$$

$$, \ldots, N,$$

$$x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau) \quad \underline{\quad} \quad {}_{q-1} f d\tau,$$

$$\text{for } t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots, N.$$

$^{c}D^{q}$

$\Leftrightarrow \{\, x(0) = _0 + x(xt) = _0,\ xf'(0) = (t,\ x(t\bar{x}))_0,,\ t \in x_0(0,\ \bar{x},_0 T \in],$
R

$$\Leftrightarrow \lim \quad (t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau) \qquad \{\text{system}(1.1)\}\ . \quad g_k q-1 d\tau$$

for $t \in (t_k, s_k]$, forall $k \in \{1, 2, ..., N\}$ Therefore, Eq. (3.11) satisfies all conditions of system (1.1), and it is a solution of system (1.1).

Remark 3.1. $\int_{s_k}^t (t-\tau)^{q-1} f d\tau (k = \ {}_{(}{}^{1}{}_{q)} 1, 2, ..., N)$ is a key Γ partoftheapproximatesolution $\tilde{x}(t)$,anditisnotincluded in Eq. (3.11). Therefore, Eq. (3.11) is a **particular solution** of (1.1).

Theorem 3.1. Let ξ_k and ζ_k (here $k = 1, 2, ..., N$) be someconstants.System(1.1)isequivalentwiththeintegral equation

$$\begin{cases} x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau \text{ for } t \in (0, t_1], \\ g_k \qquad t \in (t_k, s_k],\ k = 1, 2 (t, x(t)) \text{ for } , ..., N, \end{cases}$$

$$x(t) = \begin{cases} g_k(s_k, x(s_k)) - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \\ +(t - s_k) \left[g_k'(s_k, x(s_k)) - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] \\ + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau \\ + \{ \xi_k \left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 \quad s_k \right. \\ \left. - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau \right] \\ + \zeta_k \ g_k'(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau [\ ]\} \\ \times \{ \frac{1}{\Gamma(q)} [\int_0^{s_k} (s_k - \tau])^{q-1} f d\tau + \int_{s_k}^t (t-\tau)^{q-1} f d\tau \} \\ - \int^t (t-\tau)^{q-1} f d\tau + \frac{(t - s_k)}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau_0 \\ \qquad \text{for } t \in (s_k, t_{k+1}],\ k = 1, 2, ..., N. \end{cases} \tag{3.12}$$

provided that the integral in (3.12) exists.

Proof. ‘Sufficiency’; the solution of (1.1) for $t \in (0, t_1]$ satisfies

$$x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau \textit{ for } t \in (0, t_1], \tag{3.13}$$

and $x(t) = g_1(t, x(t))$ *for* $t \in (t_1, s_1]$.

For $t \in (s_1, t_2]$, the approximate solution of (1.1) is given by

$$\tilde{x}(t) = g_1(s_1, x(s_1)) + (t - s_1) g_1'(s_1, x(s_1)) + \frac{1}{\Gamma(q)} \int_{s_1}^t (t-\tau)^{q-1} f d\tau \text{ for } t \in (s_1, t_2]. \tag{3.14}$$

Let $e_1(t) = x(t) - \tilde{x}(t)$ for $t \in (s_1, t_2]$. Moreover, by the particular solution (3.11), the exact solution $x(t)$ of system (1.1) satisfies

$$\lim_{\substack{\left[g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau\right] \to 0 \\ \left[g_k'(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau\right] \to 0}} , \quad x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau \textit{ for } t \in (s_1, t_2].$$

Thus,

$$\lim_{\substack{\left[g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau\right] \to 0, \\ \left[g_1'(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau\right] \to 0}} e_1(t) = \lim_{\substack{\left[g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau\right] \to 0 \\ \left[g_1'(s_1, x(s_1)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau\right] \to 0}} \{x(t) - \tilde{x}(t)\}$$

$$= \frac{1}{\Gamma(q)} \left[\int_0^t (t-\tau)^{q-1} f d\tau - \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau - \int_{s_1}^t (t-\tau)^{q-1} f d\tau \right] - \frac{t - s_1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau.$$

,

(3.15)

This means $e_1(t)$ is connected with $\lim \left[g_1(s_1,x(s_1))-x_0-\bar{x}_0 s_1-\frac{1}{\Gamma(q)}\int_0^{s_1}(s_1-\tau)^{q-1}fd\tau\right]\to 0, \left[g_1'(s_1,x(s_1))-\bar{x}_0-\frac{1}{\Gamma(q-1)}\int_0^{s_1}(s_1-\tau)^{q-2}fd\tau\right]\to 0$ $e_1(t)$, $g_1(s_1,x(s_1))-x_0-\bar{x}_0 s_1-\frac{1}{\Gamma(q)}\int_0^{s_1}(s_1-\tau)^{q-1}$ $fd\tau$ and $g_1'(s_1,x(s_1))-\bar{x}_0-\frac{1}{\Gamma(q-1)}\int_0^{s_1}(s_1-\tau)_{q-2}fd\tau$. Therefore, suppose

$$_1(t)= \chi\,\big(g_1(s_1,x(s_1))-x_0-\bar{x}_0 s_1-\frac{1}{\Gamma(q)}\int_0^{s_1}(s_1-\tau)^{q-1} \qquad e \qquad fd\tau,$$

$$'_1(s_1,x(s_1))-\bar{x}_0-\frac{1}{\Gamma(q-1)}\int_0^{s_1}(s_1-\tau)^{q-2} \quad \big)\big)\big\{ \frac{1}{\Gamma(q)}\Big[\int_0^t(t-\tau)^{q-1}fd\tau \qquad g$$

$$-\int_0^{s_1}(s_1-\tau)^{q-1}fd\tau-\int_{s_1}^t(t-\tau)^{q-1} \quad -\frac{t-s_1}{\Gamma(q-1)}\int_0^{s_1}(s_1-\tau)^{q-2}fd\tau\big\} \qquad fd\tau$$

(3.16)

$$fd\tau\big] \qquad for\ t\in(s_1,t_2].$$

where $\chi(\cdot\,,\cdot)$ is an undetermined function with $\chi(0,0)=1$. Thus,

$$(t)=\tilde{x}(t)+e_1(t)=g_1(s_1,x(s_1))-\frac{1}{\Gamma(q)}\int_0^{s_1}(s_1-\tau)^{q-1}fd\tau+(t-s_1)\left[g_1'(s_1,x(s_1))-\frac{1}{\Gamma(q-1)}\int_0^t(t-\tau)^{q-2}fd\tau\right]$$

x ——

$$_1(s_1,x(s_1))-x_0-\bar{x}_0 s_1-\frac{1}{\Gamma(q)}\int_0^{s_1}(s_1-\tau)^{q-1}fd\tau$$

$$'_1(s_1,x(s_1))-\bar{x}_0-\frac{1}{\Gamma(q-1)}\int_0^{s_1}(s_1-\tau) \quad \big)\big)\big]\big\{ \frac{1}{\Gamma(q)}\Big[\int_0^{s_1}(s_1-\tau)$$

$$+\int_{s_1}^t(t-\tau)^{q-1}fd\tau-\int_0^t(t-\tau)^{q-1} \quad +\frac{t-s_1}{\Gamma(q-1)}\int_0^{s_1}(s_1-\tau)^{q-2}fd\tau\big\}$$

$$+\frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}fd\tau+\Big[1-\chi\Big(g \qquad ,$$

$$q-2fd\tau \qquad q-1fd\tau \qquad (3.17)$$

g

$$fd\tau\big] \qquad \text{for } t\in(s_1,t_2].$$

On the other hand, letting $t_1\to s_1$, we get

$$\Big\{\ {}^{C}D_{0+}^{q}x(t)=f(t,x(t)),\ q\in(1,2),\ t\in(s_k,t_{k+1}],\ k=0,1,\ \lim x(t)$$

$$=g_1(t,x(t)),\ t\in(t_1,s_1],\ {}_{t_1\to s_1}\ x(0)={}_0x'(0)=\bar{x}_0,\ x_0,\bar{x}_0\in \mathrm{R}.\ x\,,$$

$$\Big\{\ {}^{C}D_{0+}^{q}x(t)=f(t,x(t)),\qquad q\in(1,2),\ t\in(s_k,t_{k+1}],\ k=0,1,$$

$$=\ x(t_1)=x(s_1)=g_1(s_1,x(s_1)),$$

$$x'(t_1)=x'(s_1)=g_1'(s_1,x(s_1)), \qquad (3.18)$$

$$(0)=\ ,\ '(0)=\bar{}\ ,\ \ ,\bar{}\ \ \mathrm{R} \quad x \quad x_0\,x \quad x_0 \quad x_0\,x_0 \in \quad .$$

$$\Big\{\ {}^{C}D_{0+}^{q}x(t)=f(t,x(t)),\qquad q\in(1,2),\ t\in(s_k,t_{k+1}],\ k=0,1,$$

$$x'(s_1^+)-x'(s_1^-)=g_1'(s_1,x(s_1))-\bar{x}_0-\frac{1}{\Gamma(q-1)}\int_0^{s_1}(s_1-\tau)^{q-2}f(\tau,(x(\tau))$$

$$\Leftrightarrow\quad x(s_1^+)-x(s_1^-)=g_1(s_1,x(s_1))-x_0-\bar{x}_0 s_1-\frac{1}{\Gamma(q)}\int_0^{s_1}(s_1-\tau)^{q-1}f(\tau,(x(\tau))\,d\tau,$$

$$d\tau,\quad x(0)=x_0,\ x'(0)=\bar{x}_0,\ x_0,\bar{x}_0\in \mathrm{R}.$$

Using **Lemma 2.4** for system (3.18), we get $1-\chi(y,z)=\xi_1 y+\zeta_1 z$ for $\forall y,z\in \mathrm{R}$, here ξ_1 and ζ_1 are two constants. Thus,

$$x(t) = g_1(s_1, x(s_1)) - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau + (t - s_1) \left[g_1'(s_1, x(s_1)) - \frac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau \right]$$

$$+ \frac{1}{\Gamma(q)} \int_0^{t} (t - \tau)^{q-1} f d\tau + \quad \xi_1 \quad g_1(s_1, x(s_1)) - x_0 - \bar{x}_0 s_1 - \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - \tau)^{q-1} f d\tau$$

$$\{ \quad [\qquad \qquad]$$

$$[\, '(s_1, x(s_1)) - \bar{x}_0 - \tfrac{1}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)_{q-2} f d\tau]\}\{ \tfrac{1}{\Gamma(q)} [\int_0^{s_1} (s_1 - \tau)_{q-1} f d\tau \quad (3.19) \quad +\zeta_1\, g_1$$

$$+ \int_{s_1}^{t} (t - \tau)^{q-1} f d\tau - \int_0^{t} (t - \tau)^{q-1} f d\tau] + \frac{(t - s_1)}{\Gamma(q-1)} \int_0^{s_1} (s_1 - \tau)^{q-2} f d\tau \Big\} \underline{\qquad} \text{ for } t \in (s_1, t_2].$$

and $x(t) = g_2(t, x(t))$ for $t \in (t_2, s_2]$.

Next, for $t \in (s_k, t_{k+1}]$ (here $k \in \{1, 2, ..., N\}$), the approximate solution of (1.1) is provided by

$$\tilde{x}(t) = g_k(s_k, x(s_k)) + (t - s_k) g_k'(s_k, x(s_k)) + \frac{1}{\Gamma(q)} \int_{s_k}^{t} (t - \tau)^{q-1} f d\tau \text{ for } t \in (s_k, t_{k+1}]. \quad (3.20)$$

Let $e_k(t) = x(t) - \tilde{x}(t)$ for $t \in (s_k, t_{k+1}]$. Moreover, by the particular solution (3.11), the exact solution $x(t)$ of system (1.1) satisfies

$$\lim_{\substack{\left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau\right] \to 0, \\ \left[g_k'(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau\right] \to 0,}} x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^{t} (t - \tau)^{q-1} f d\tau \text{ for } t \in (s_k, t_{k+1}].$$

Thus,

$$\lim_{\substack{\left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau\right] \to 0, \\ \left[g_k'(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau\right] \to 0,}} e_k(t)$$

$$= \lim_{\substack{\left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau\right] \to 0 \\ \left[g_k'(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau\right] \to 0,}} \{x(t) - \bar{x}(t)\},$$

$$= \frac{-1}{\Gamma(q)} \left[\int_0^{s_k} (s_k - \tau)^{q-1} f d\tau + \int_{s_k}^{t} (t - \tau)^{q-1} f d\tau - \int_0^{t} (t - \tau)^{q-1} f d\tau \right] - \frac{t - s_k}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau, \quad (3.21)$$

Similarly to (3.16), suppose

$$\begin{aligned} e_k(t) = \kappa \Big(& g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau, \\ & g_k'(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \Big) \Big\{ \frac{1}{\Gamma(q)} \Big[\int_0^{t} (t - \tau)_{q-1} f d\tau \\ & - \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau - \int_{s_k}^{t} (t - \tau)^{q-1} f d\tau \Big] - \frac{t - s_k}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \Big\} \text{ for } t \in (s_k, t_{k+1}]. \end{aligned} \quad (3.21)$$

where $\kappa(\cdot\,, \cdot)$ is an undetermined function with $\kappa(0, 0) = 1$. Thus,

$$\begin{aligned} x(t) = \tilde{x}(t) + e_k(t) = {} & g_k(s_k, x(s_k)) - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} \underline{\quad} f d\tau \\ & + (t - s_k) \left[g_k'(s_k, x(s_k)) - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \right] + \frac{1}{\Gamma(q)} \int_0^{t} (t - \tau)_{q-1} f d\tau \\ & + \Big[1 - \kappa \Big(g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k - \tau)^{q-1} f d\tau, \end{aligned} \quad (3.22)$$

$$g_k'(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k - \tau)^{q-2} f d\tau \Big) \Big] \Big\{ \frac{1}{\Gamma(q)} \Big[\int_0^{s_k} (s_k - \tau)_{q-1} f d\tau$$

$$+ \int_{s_k}^t (t-\tau)^{q-1} f d\tau - \int_0^t (t-\tau)^{q-1} f d\tau \Big] + \frac{t-s_k}{\Gamma(q-1)} \int_0^{s_k} (s_k-\tau)^{q-2} f d\tau \Big\} \quad \text{for } t \in (s_k, t_{k+1}].$$

Moreover, considering a special case $g_i(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)_{q-1} f d\tau$ for all $i \in \{1, 2, ..., k-1\}$ and $t_k \to s_k$ in system (1.1), we have

$$\begin{cases} x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau) \\ x(t) = g_k(t, x(t)), \quad t \in (t_k, s_k], \\ x(0) = x_0, x'(0) = \bar{x}_0, \quad x_0, {}^{-}{}_0 \in \mathrm{R} \end{cases} \quad \lim \quad {}^C D_{0+}{}^q x(t) = f(t, x(t)), \quad q \quad \in$$

$(1,{}^q2)^{-1},ft(\tau\in,(xs(_i\tau,))t_id\tau{+1}],, it= 0\in (,t_i1,, ...,s_i], ik= 1, \quad , 2, ..., k-1, t_k\to s_k$

x .

$$\begin{cases} {}^C D_{0+}{}^q x(t) = f(t, x(t)), \quad q \in (1, 2), \quad t \in (s_i, t_{i+1}], \quad i = 0, 1, ..., k, \\ x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau, x(\tau)) \quad d\tau, t \in (t_i, s_i], i = 1, 2, ..., k-1, \end{cases}$$

$= {}^x('(ss_{kk}) =) = {}^x x('t(_kt) =_k) = gg_k('_ks(_ks,_k, {}^x x(s(_ks))_k)),, \quad (3.23)\ x$

$x(0) = x_0, x'(0) = x^{-}{}_0, \quad x_0, x^{-}{}_0 \in \mathrm{R}.$

$${}^C D_{0+}{}^q x(t) = f(t, x(t)), \quad q \in (1, 2), t \in (s_i, t_{i+1}], i = 0, 1, ..., k,$$

$$(s_k^+) - x(s_k^-) = g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k-\tau)^{q-1}$$

$$x(t) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau, x(\tau)) d\tau, t \in (t_i, s_i], i = 1, 2, ..., k-1,$$

$\Leftrightarrow \qquad xf(\tau, x(\tau))d\tau,$

$$x'(s_k^+) - x'(s_k^-) = g_k'(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k-\tau)^{q-2} f(\tau, x(\tau)) d\tau,$$

$$x(0) = x_0, x'(0) = \bar{x}_0, \quad x_0, {}^{-}{}_0 \in \mathrm{R}_x.$$

Using **Lemma 2.4** and Eq. (3.23) for (3.24), we have $1- \kappa(y, z) = \xi_k y + \zeta_k z$ for $\forall y, z \in \mathrm{R}$, here ξ_k and ζ_k are two constants. Thus,

$$x(t) = g_k(s_k, x(s_k)) - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k-\tau)^{q-1} f d\tau + (t-s_k) \Big[g_k'(s_k, x(s_k)) - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k-\tau)^{q-2} f d\tau \Big]$$

$$+ \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f d\tau + \Big\{ \xi_k \Big[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k-\tau)^{q-1} f d\tau \Big]$$

$$+\zeta_k \Big[g_k'(s_k, x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)} \int_0^{s_k} (s_k-\tau)^{q-2} f d\tau \Big] \Big\} \Big\{ \frac{1}{\Gamma(q)} \Big[\int_0^{s_k} (s_k-\tau)_{q-1} f d\tau$$

$$+ \int_{s_k}^t (t-\tau)^{q-1} f d\tau - \int_0^t (t-\tau)^{q-1} f d\tau \Big] + \frac{(t-s_k)}{\Gamma(q-1)} \int_0^{s_k} (s_k-\tau)^{q-2} f d\tau \Big\}, \quad \text{for } t \in (s_k, t_{k+1}].$$

'Necessity'; taking the fractional derivative to Eq. (3.12) for $t \in (s_k, t_{k+1}]$ (here $k = 1, 2, ..., N$), we get

$$cD_{0+q}\, x_{q-1} f d\tau \quad (t)\Big|_{t\in(s_k, t_{k+1}]} = {}^C D_{0+}^q \Big\{ g_k(s_k, x(s_k)) - \frac{1}{\Gamma(q)} \int_0^{s_k} (s_k-\tau)^{q-1} f d\tau + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)$$

$t \quad s_k \quad g_k \; s_k \; x \; s_k \quad \overline{\Gamma(q-1)} \quad 0^{k} \; s_k \quad \tau^{q} \quad fd\tau$ +(−)[′ (, ( )) − 1 ∫ s (−) −2]

$$+\left\{\xi_k\left[g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)}\int_0^{s_k}(s_k-\tau)^{q-1}fd\tau\right]\right.$$

$k \quad ' \quad - \quad 1 \quad s_k \quad q-2 \quad 1 \quad [\int^{s_k} (\; - \;)$

$+\zeta\,[g_k(s_k, x(s_k)) - x_0 - \cancel{\Gamma(q-1)}\int_0 (s_k-\tau) \quad fd\tau]\}\{\, \cancel{\Gamma(q)}\text{———}0 \quad s_k \quad \tau^{\,q-1}fd\tau$

$$+\int_{s_k}^{t}(t-\tau)^{q-1}fd\tau - \int_0^t(t-\tau)^{q-1}fd\tau\Big] + \frac{(t-s_k)}{\Gamma(q-1)}\int_0^{s_k}(s_k-\tau)^{q-2}fd\tau\Big\}\Big\} \quad \text{———}$$

$t\in(s_k,t_{k+1}]$

$$= f(t,x(t))\big|_{t\in(s_k,t_{k+1}]} + \frac{1}{\Gamma(q)}\left\{\xi_k\left[g_k(s_k,x(s_k)) - x_0 - \bar{x}_0 s_k - \frac{1}{\Gamma(q)}\int_0^{s_k}(s_k-\tau)^{q-1}fd\tau\right]\right.$$

$$\left.+\zeta_k\left[g_k'(s_k,x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)}\int_0^{s_k}(s_k-\tau)^{q-2}fd\tau\right]\right\}$$

$$\times\left[{}^{C}D^{q}_{s_k+}\left(\int_{s_k}^{t}(t-\tau)^{q-1}f(\tau,x(\tau))d\tau\right) - {}^{C}D^{q}_{0+}\left(\int_0^t(t-\tau)^{q-1}f(\tau,x(\tau))d\tau\right)\right]_{t\in(s_k,t_{k+1}]}$$

$$= f(t,x(t))\big|_{t\in(s_k,t_{k+1}]} + \quad \xi_k \quad g_k(s_k,x(s_k)) - x_0 - x_0 s_k - \overline{\Gamma(q)}\int_0^{s_k}(s_k-\tau)^{q} \quad fd\tau_{\{[} \quad - \quad 1 \quad -1$$

]

$$+\zeta_k\,[g'_k(s_k,\ x(s_k)) - \bar{x}_0 - \cancel{\Gamma(q1-1)}\int_0^{s_k}(s_k-\tau)^{q-2}fd\tau]\}[f(t,\ x(t))\,\big|\,\big|_{t>s_k} - f(t,\ x(t))\,\big|\,\big|_{t>0}]_{t\in(s_k,t_{k+1}]} = f(t,$$

$$x(t))\,\big|\,\big|_{t\in(s_k,t_{k+1}]}\,.$$

Therefore, Eq. (3.12) satisfies the condition (1.1a). Next, Eq. (3.12) also satisfies the conditions (1.1b) and (1.1c). Furthermore, by Eq. (3.12), we obtain

$\lim \quad x(t)$

$g_k(t,x(t)) = x_0 + \bar{x}_0 \quad t+ \quad 1 \quad \int^t (t-\tau) \quad \cancel{\Gamma(q)}\text{——}0 \quad {}^{q}{-1}d\tau$

for $t \in (t_k, s_k]$, forall $k \in \{1, 2, ..., N\}$ $\Big($ $x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}$ $fd\tau$ for $t \in (0, t_1]$,

$\Big|$ $g_k(t, x(t))$ for $t \in (t_k, s_k]$, $k = 1, 2, ..., N$,

$||$

$|$ $+(t-s_k)\;\; g_k'(s_k,x(s_k)) - \frac{1}{\Gamma(q-1)}\int_0^{s_k}(s_k-\tau)^{q-2}fd\tau$

$\Big| \; g_k \quad k \quad k \quad \frac{1}{\Gamma(q)} \; {}^{s_k}_{0} \quad k \quad q-1 \quad \frac{1}{\Gamma(q)} \; {}^{t}_{0}$

$\Big[\qquad \Big]$ $(s\ , x(s\ )) - \quad \text{——}\int \quad (s-\tau) \quad fd\tau + \quad \text{——}\int (t$

$-\,\tau)^{q-1}fd\tau$

$=g_k(t,x(t)) = x_0 + \bar{x} \quad \frac{1}{q} \quad t \quad q-1 \quad \Big\} \quad +\Big\{\xi_k\Big[{}_0\lim_{t+\,\Gamma(\,)}\int_0 (t-\tau) \quad d\tau \; || \; g_k(s_k, x(s_k)) - x_0 - \bar{x}_0 s_k$

$-\frac{1}{\Gamma(q)}\int_0^{s_k}(s_k-\tau)^{q-1}fd\tau\Big]$ $\Big|$ $\times\left\{\frac{1}{\cancel{\Gamma(q)}}\left[\int_0^{s_k}(s_k-\tau)^{q-1}fd\tau + \int_{s_k}^{t}(t-\tau)^{q-1}fd\tau - \int_0^t(t-\tau)^{q-1}fd\tau\right]\right.$

for $t \in (t_k, s_k]$, forall $k \in \{$

$1, 2, \ldots, N\}$ $\Big|$ $+\zeta_k\left[g_k'(s_k,x(s_k)) - \bar{x}_0 - \frac{1}{\Gamma(q-1)}\int_0^{s_k}(s_k-\tau)^{q-2}fd\tau\right]\Big\}$———

$||\Big($ $+\frac{(t-s_k)}{\Gamma(q-1)}\int_0^{s_k}(s_k-\tau)^{q-2}$ $fd\tau\}$ for $t \in (s_k, t_{k+1}]$, $k = 1, 2, ..., N$.

$\Big($ $x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)}\int_0^t(t-\tau)_{q-1}fd\tau$, for $t \in (0, t_1]$,

$= \Big\{$ $x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)}\int_0^t(t-\tau)_{q-1}fd\tau$, for $t \in (t_k, s_k]$, $k = 1, 2, ..., N$, $||\Big($ $x_0 +$

$\bar{x}_0 t + \frac{1}{\Gamma(q)}\int_0^t(t-\tau)^{q-1}fd\tau$, for $t \in (s_k, t_{k+1}]$, $k = 1, 2, ..., N$. ⇔ {system(3.4)}

Thus,

$$\{\text{Eq.}(3.12)\} \Leftrightarrow \{\text{system}(1.1)\} \quad \lim g_k(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} d\tau \text{ for } t \in (t_k, s_k], \text{ forall } k \in \{1, 2, \ldots, N\}$$

$$\lim g_k(t, x(t)) = x_0 + \bar{x}_0 t + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} d\tau \, g_k \text{ for } t \in (t_k, s_k], \text{ forall } k \in \{1, 2, \ldots, N\}$$

So, Eq. (3.12) satisfies all conditions of system (1.1).

By “Sufficiency” and “Necessity”, system (1.1) is equivalent to Eq. (3.12). The proof is completed. ☐

4 Example

Example 1. Let us consider the general solution of the impulsive fractional system

$$\begin{cases} {}^{C}D_{0+}^{\frac{5}{4}} x(t) = t, & t \in (0, \frac{\pi}{4}] \cup (\frac{\pi}{2}, \pi], \\ x(t) = \sin t, & t \in (\frac{\pi}{4}, \frac{\pi}{2}], \\ x(0) = 1, \quad x'(0) = 1. \end{cases} \tag{4.1}$$

By **Theorem 3.1**, system (4.1) has a general solution

$$x(t) = \begin{cases} 1 + t + \frac{t^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} & \text{for } t \in (0, \frac{\pi}{4}], \\ \sin t & \text{for } t \in (\frac{\pi}{4}, \frac{\pi}{2}], \\ 1 + \frac{t^{\frac{9}{4}}\big|_{t>0} - (\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} - \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} (t - \frac{\pi}{2})\big|_{t>\frac{\pi}{2}} - \left\{ \xi \left[\frac{\pi}{2} + \frac{(\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} \right] + \zeta \left[1 + \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} \right] \right\} & \\ \quad \times \left\{ \frac{1}{\Gamma(\frac{13}{4})} \left[(\frac{\pi}{2})^{\frac{9}{4}} + (t - \frac{\pi}{2})^{\frac{5}{4}} (t + \frac{5\pi}{8})\Big|_{t>\frac{\pi}{2}} - t^{\frac{9}{4}}\Big|_{t>0} \right] + \frac{(\frac{\pi}{2})^{\frac{5}{4}} (t - \frac{\pi}{2})\big|_{t>\frac{\pi}{2}}}{\Gamma(\frac{9}{4})} \right\} & \text{for } t \in (\frac{\pi}{2}, \pi]. \end{cases} \tag{4.2}$$

where ξ and ζ are two constants.

Eq. (4.2) for $t \in (0, \frac{\pi}{4}]$ satisfies fractional derivative conditioninsystem(4.1)by**Lemma2.3**,andfor $t \in (\frac{\pi}{2}, \pi]$, we have

$$\begin{aligned} {}^{C}D_{0+}^{\frac{5}{4}} x(t)\Big|_{t\in(\frac{\pi}{2},\pi]} &= {}^{C}D_{0+}^{\frac{5}{4}} \left\{ 1 + \frac{t^{\frac{9}{4}}\big|_{t>0} - (\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} - \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} (t - \frac{\pi}{2})\Big|_{t>\frac{\pi}{2}} \right. \\ &\quad - \left\{ \xi \left[\frac{\pi}{2} + \frac{(\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} \right] + \zeta \left[1 + \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} \right] \right\} \\ &\quad \times \left\{ \frac{1}{\Gamma(\frac{13}{4})} \left[(\frac{\pi}{2})^{\frac{9}{4}} + (t - \frac{\pi}{2})^{\frac{5}{4}} (t + \frac{5\pi}{8})\Big|_{t>\frac{\pi}{2}} - t^{\frac{9}{4}}\Big|_{t>0} \right] \right. \\ &\quad \left. \left. + \frac{(\frac{\pi}{2})^{\frac{5}{4}} (t - \frac{\pi}{2})\big|_{t>\frac{\pi}{2}}}{\Gamma(\frac{9}{4})} \right\} \right\}_{t\in(\frac{\pi}{2},\pi]} \end{aligned}$$

$$= {}^{C}D_{0+}^{\frac{5}{4}} \left\{ \frac{t^{\frac{9}{4}}\Big|_{t>0}}{\Gamma(\frac{13}{4})} - \frac{1}{\Gamma(\frac{13}{4})} \left\{ \xi \left[\frac{\pi}{2} + \frac{(\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} \right] \right.\right.$$

$$\left.\left. +\zeta \left[1 + \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} \right] \right\} \left[(t-\frac{\pi}{2})^{\frac{5}{4}}(t+\frac{5\pi}{8})\Big|_{t>\frac{\pi}{2}} - t^{\frac{9}{4}}\Big|_{t>0} \right] \right\}_{t\in(\frac{\pi}{2},\pi]}$$

$$= \left\{ t|_{t>0} - \frac{1}{\Gamma(\frac{13}{4})} \left\{ \xi \left[\frac{\pi}{2} + \frac{(\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} \right] + \zeta \left[1 + \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} \right] \right\} \right.$$

$$\left. \left[{}^{C}D_{\frac{\pi}{2}+}^{\frac{5}{4}} \left((t-\frac{\pi}{2})^{\frac{5}{4}}(t+\frac{5\pi}{8})\Big|_{t>\frac{\pi}{2}} \right) - {}^{C}D_{0+}^{\frac{5}{4}} \left(t^{\frac{9}{4}}\Big|_{t>0} \right) \right] \right\}_{t\in(\frac{\pi}{2},\pi]}$$

$$= \left\{ t|_{t>0} - \left\{ \xi \left[\frac{\pi}{2} + \frac{(\frac{\pi}{2})^{\frac{9}{4}}}{\Gamma(\frac{13}{4})} \right] + \zeta \left[1 + \frac{(\frac{\pi}{2})^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} \right] \right\} \right.$$

$$\left. \times \left[t|_{t>\frac{\pi}{2}} - t|_{t>0} \right] \right\}_{t\in(\frac{\pi}{2},\pi]} = t|_{t\in(\frac{\pi}{2},\pi]} .$$

Therefore, Eq. (4.2) (for $t \in (0, \frac{\pi}{4}] \cup (\frac{\pi}{2}, \pi]$) satisfies fractional derivative condition in system (4.1). Meanwhile, Eq. (4.2)satisfiesthenon-instantaneousimpulsesconditionin system (4.1). Thus, Eq. (4.2) is the general solution of (4.1).

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