Interval-Valued Multifunctions and the Riemann-Lebesgue Integral

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Abstract: We study Riemann-Lebesgue integrability for interval-valued multifunctions relative to an interval-valued set multifunction. Some classic properties of the *RL* integral, such as monotonicity, order continuity, bounded variation, convergence are obtained. An application of interval-valued multifunctions to image processing is given for the purpose of illustration; an example is given in case of fractal image coding for image compression, and for edge detection algorithm. In these contexts, the image modelization as an interval valued multifunction is crucial since allows to take into account the presence of quantization errors (such as the so-called round-off error) in the discretization process of a real world analogue visual signal into a digital discrete one.

Keywords: Riemann-Lebesgue integral; interval valued (set) multifunction; non-additive set function; image processing

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1. Introduction

The theory of multifunctions is an important field of research. Since interval arithmetic, introduced by Moore in [1], it appears a natural option for handling the uncertainty in data and in sensor measurements, particular attention was addressed to the study of interval-valued multifunctions and multimeasures because of their applications in statistics, biology, theory of games, economics, social sciences and software, to keep track of rounding errors in calculations and of uncertainties in the knowledge of the exact values of physical and technical parameters (see for example [2–5]). In fact, since the uncertainty of information could affect an expert's opinion, the ability to consider the uncertainty information during the process could be very important, see for example [2–4,6–11] and the references therein.

However, in some recent papers, interval-valued multifunctions have been applied also to some new directions, involving signal and image processing. Digital images are in fact the result of a discretization of the reality; namely sampled version of a continuous signal. Hence, there are different sources of uncertainty and ambiguity to be considered when performing image processing tasks, see for example [12,13]. For instance, the applications of fractal image coding for image compression [14,15] is one of the topic in which interval-valued multifunctions have been applied. Clearly, image compression

techniques [16] are very useful in order to speed up the processes of digital image transmission and to improve the efficiency of image storage for high dimensional databases [17]. Further, applications of interval-valued multifunctions to the implementation of edge detection algorithms can also be found (see e.g., [13,18]).

In the literature several methods of integration for functions and multifunctions have been studied extending the Riemann and Lebesgue integrals. In this framework a generalization of Riemann sums was given in [19–37] while another generalization is due to Kadets and Tseytlin [38], who introduced the absolute Riemann-Lebesgue $|RL|$ and unconditional Riemann-Lebesgue RL integrability, for Banach valued functions with respect to countably additive measures. They proved that in finite measure space, the Bochner integrability implies $|RL|$ integrability which is stronger than RL integrability that implies Pettis integrability. Regarding this last extension contributions are given also in [21,23,34,39].

In the last decade the study of non-additive set functions and multifunctions has recently received a wide recognition, (see also [3,9,10,40–46]). In this paper, motivated by the large number of fields in which the interval-valued multifunction can be applied, we introduce a new type of integral of an interval-valued multifunction G with respect to an interval-valued submeasure M with respect to the weak interval order relation introduced in [4] by Guo and Zhang. Although the construction procedure of the integral is similar to the one given in [34,38,39], the integral proposed is a generalization of it since we are concerned with the study of a Riemann-Lebesgue set-valued integrand with respect to an arbitrary interval-valued set function, not necessarily countably additive. So the novelty of this construction concerns not only the codomain of the integrands but also the non-additivity of the measure with respect to which they are integrated. The main results on this subject are Theorem 1, in which the additivity of the integral is proved even if the pair (G, M) does not satisfy this property; the monotonicity and the order continuity are established in Theorems 2 and 4 and a convergent result given in Theorem 5.

The paper is organized as follows: in Section 2 the basic concepts and terminology are introduced together with some remarks. In Section 3 we introduce the RL-integral of an interval-valued multifunction with respect to an interval valued subadditive multifunction and we provide a comprehensive treatment of the integration theory together with a comparison with other integrals defined in the same setting (Remark 8). An example of an application in image processing is given in Section 3.1. The applications concerning image processing discussed in the present paper is given for the purpose of illustration and is new. The main reason for which we discuss the above application is to provide examples and justifications of the uses of interval-valued multifunctions to concrete applications in Image Processing. The advantage of using the notion of interval-valued multifunction in signal analysis is that this formalism allows to include in a unique framework possible uncertainty or the noise on the evaluation of an image at any given pixel.

2. Preliminaries

Let S be a nonempty at least countable set, $\mathcal{P}(S)$ the family of all subsets of S and $\mathcal{A}$ a σ-algebra of subsets of S. The symbol $\mathbb{R}_0^+$ denotes, as usual, the set of non negative real numbers.

Definition 1 ([34], Definition 2.1)**.**

(i) *A finite (countable) partition of S is a finite (countable) family of nonempty sets* $P = \{A_i\}_{i=1,\dots,n}$ $(\{A_n\}_{n\in\mathbb{N}}) \subset \mathcal{A}$ *such that* $A_i \cap A_j = \emptyset$, $i \neq j$ *and* $\bigcup_{i=1}^{n} A_i = S$ $(\bigcup_{n\in\mathbb{N}} A_n = S)$.

(ii) *If P and P^0 are two partitions of S, then P^0 is said to be finer than P, denoted by $P \leq P^0$ (or $P^0 \geq P$), if every set of P^0 is included in some set of P.*

(iii) *The common refinement of two finite or countable partitions $P = \{A_i\}$ and $P^0 = \{B_j\}$ is the partition $P \wedge P^0 = \{A_i \cap B_j\}$.*

(iv) *A countable tagged partition of S if a family $\{(B_n, s_n), n \in \mathbb{N}\}$ such that $(B_n)_n$ is a partition of S and $s_n \in B_n$ for every $n \in \mathbb{N}$.*

We denote by P the class of all the countable partitions of S and if $A \in \mathbb{A}$ is fixed, by P_A we denote the class of all the countable partitions of the set A.

Definition 2 ([34], Definition 2.2)**.** *Let $m : \mathbb{A} \to [0, +\infty)$ be a non-negative function, with $m(\emptyset) = 0$. A set $A \in \mathbb{A}$ is said to be an atom of m if $m(A) > 0$ and for every $B \in \mathbb{A}$, with $B \subset A$, it is $m(B) = 0$ or $m(A \backslash B) = 0$. m is said to be:*

(i) *monotone if $m(A) \leq m(B)$, $\forall\ A, B \in \mathbb{A}$, with $A \subseteq B$;*

(ii) *subadditive if $m(A \cup B) \leq m(A) + m(B)$, for every $A, B \in \mathbb{A}$, with $A \cap B = \emptyset$;*

(iii) *a submeasure (in the sense of Drewnowski [47]) if m is monotone and subadditive;*

(iv) *σ-subadditive if $m(A) \leq \sum_{n=0}^{+\infty} m(A_n)$, for every sequence of (pairwise disjoint) sets $(A_n)_{n\in\mathbb{N}} \subset \mathbb{A}$, with $A = \bigcup_{n=0}^{+\infty} A_n$.*

(v) *order-continuous (shortly, o-continuous) if $\lim_{n\to\infty} m(A_n) = 0$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathbb{A}$, with $A_n \searrow \emptyset$;*

(vi) *exhaustive if $\lim_{n\to\infty} m(A_n) = 0$, for every sequence of pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subset \mathbb{A}$.*

(vii) *null-additive if $m(A \cup B) = m(A)$, for every $A, B \in \mathbb{A}$, with $m(B) = 0$;*

Moreover m satisfies property (σ) if the ideal of m-zero sets is stable under countable unions (see for example [34], Definition 2.3).

We denote by the symbol $ck(\mathbb{R})$ the family of all non-empty convex compact subsets of $\mathbb{R}$, by convention, $\{0\} = [0, 0]$. We consider on $ck(\mathbb{R})$ the Minkowski addition ($A + B := \{a + b : a \in A, b \in B\}$) and the standard multiplication by scalars. $\|A\| := \sup\{|x|: x \in A\}$. d_H is the Hausdorff distance in $ck(\mathbb{R})$, while $e(A, B) = \sup\{d(x, B), x \in A\}$ and $d_H(A, B) = \max\{e(A, B), e(B, A)\}$.

$(ck(\mathbb{R}), d_H)$ is a complete metric space ([48,49]), but is not a linear space since the subtraction is not well defined.

If $A = [a, b]$ then $\|A\| = \max\{|a|, |b|\}$. Moreover

$$d_H([a, b], [c, d]) = \max\{|a - c|, |b - d|\}, \quad \forall\ a, b, c, d \in \mathbb{R}$$
$$d_H([0, a], [0, b]) = |b - a| \quad \forall\ a, b \in \mathbb{R}_0^+.$$

In the family $ck(\mathbb{R})$ the following operations are also considered, for every $a, b, c, d \in \mathbb{R}$:

(i) $[a, b] \cdot [c, d] = [ac, bd]$;

(ii) $[a, b] \subseteq [c, d]$ if and only if $c \leq a \leq b \leq d$; **(iii)** $[a,b] \preceq [c,d]$ if and only if $a \leq c$ and $b \leq d$; (weak interval order)

(iv) $[a, b] \wedge [c, d] = [\min\{a, c\}, \min\{b, d\}]$;

(v) $[a, b] \vee [c, d] = [\max\{a, c\}, \max\{b, d\}]$.

In general there is no relation between $'' \preceq ''$ **(iii)** and $'' \subseteq ''$ **(ii)**; they only coincide on the subfamily $\{[0, a], a \geq 0\}$. Let $ck(\mathbb{R}_0^+) := \{[a,b], a, b \in \mathbb{R}$ and $0 \leq a \leq b\}$.

In this paper we consider $(ck(\mathbb{R}_0^+), d_H, \preceq)$, namely the space $ck(\mathbb{R}_0^+)$ is endowed with the Hausdorff distance and the weak interval order. As a particular case of [20] (Definition 2.1) we have:

Definition 3. *Let $(a_n)_n$, $(b_n)_n$ be two sequences of real numbers so that $0 \leq a_n \leq b_n$, $\forall n \in \mathbb{N}$. The series $\sum_{n=0}^{\infty}[a_n, b_n] := \{\sum_{n=0}^{\infty} y_n : a_n \leq y_n \leq b_n, \forall n \in \mathbb{N}\}$ is called convergent if the sequence of partial sums $S_n := [\sum_{k=0}^{n} a_k, \sum_{k=0}^{n} b_k]$ is d_H-convergent to it.*

Remark 1. *It is easy to see that $\sum_{n=0}^{\infty}[a_n, b_n] = [u, v]$, with $0 \leq u \leq v < \infty$, if and only if $\sum_{n=0}^{\infty} a_n = u$ and $\sum_{n=0}^{\infty} b_n = v$.*

We recall the following definition for the integrable Banach-valued functions $f : S \to X$ with respect to non-negative measures given in [38,39]:

Definition 4. *A function f is called unconditional Riemann-Lebesgue (RL) m-integrable (on S) if there exists $b \in X$ such that for every $\varepsilon > 0$, there exists a countable partition P_ε of S, so that for every countable partition $P = \{A_n\}_{n \in \mathbb{N}}$ of S with $P \geq P_\varepsilon$, f is bounded on every A_n, with $m(A_n) > 0$ and for every $t_n \in A_n$, $n \in \mathbb{N}$, the series $\sum_{n=0}^{+\infty} f(t_n)m(A_n)$ is unconditional convergent and*

$$\| \sum_{n=0}^{+\infty} f(t_n)m(A_n) - b\| < \varepsilon.$$

The vector b (necessarily unique) is called *the Riemann-Lebesgue m-integral of f on S* and it is denoted by $(RL)\int_S f dm$. The *RL* definition of the integrability on a subset $A \in \mathcal{A}$ is given in the classical manner.

Remark 2. *We remember that, in the countably additive case, unconditional RL-integrability is stronger than Birkhoff integrability (in the sense of Fremlin), see Ref. [23] and the references therein; while the notion of unconditional Riemann-Lebesgue integrability coincides with Birkhoff's one given in [21] (Definition 1, Proposition 2.6 and note at p. 8).*

For the properties of this integral with respect to a submeasure we refer to the results given in [34]. Moreover we have that

Proposition 1. *Let $g_n : S \to \mathbb{R}_0^+$ be an increasing sequence of bounded RL integrable function with respect to a submeasure $\mu : \mathcal{A} \to \mathbb{R}_0^+$ of bounded variation. If there exists a $g : S \to \mathbb{R}_0^+$ such that*

(a) $g_n \to g$ *uniformly,*

(b) $\sup_n (RL)\int_S g_n d\mu < +\infty$,

then g is RL integrable with respect to μ and

$$\lim_{n\to\infty} (RL)\int_S g_n\, d\mu = (RL)\int_S g\, d\mu.$$

Proof. Since $g_n \uparrow$, by the monotonicity we have that $(RL)\int_S g_n d\mu \uparrow$ so $\sup_n (RL)\int_S g_n\, d\mu = \lim_{n\to\infty} (RL)\int_S g_n\, d\mu = u \in \mathbb{R}_0^+$. Thanks to uniform convergence g is bounded; let $L > 0$ an upper bound for g.

Let $\varepsilon > 0$ be fixed and consider $k(\varepsilon) \in \mathbb{N}$ be such that

$$|g(t) - g_{k(\varepsilon)}(t)| < \frac{\varepsilon}{3\mu(S)} \quad \forall\, t \in S, \quad \text{and}$$

$$\left|(RL)\int_S g_{k(\varepsilon)} d\mu - u\right| < \frac{\varepsilon}{3}.$$

For every countable partition $P := (A_n)_n$ finer than $P_{\varepsilon/3,k(\varepsilon)}$ (the one that verifies Definition 4 for $g_{k(\varepsilon)}$) and for every $t_n \in A_n$ we have that $\sum_{n=0}^{+\infty} g(t_n)\mu(A_n)$ converges, since μ is of bounded variation.

In fact $g(t_n)\mu(A_n) \leq L\mu(A_n)$ for every $n \in \mathbb{N}$ and, for every $k \in \mathbb{N}$, it is $0 \leq \sum_{n=0}^{k} \mu(A_n) \leq \bar{\mu}(S)$. Moreover

$$\left|\sum_{n=0}^{+\infty} g(t_n)\mu(A_n) - u\right| \leq \left|\sum_{n=0}^{+\infty} g(t_n)\mu(A_n) - \sum_{n=0}^{+\infty} g_{k(\varepsilon)}(t_n)\mu(A_n)\right| + \left|\sum_{n=0}^{+\infty} g_{k(\varepsilon)}(t_n)\mu(A_n) - (RL)\int_S g_{k(\varepsilon)} d\mu\right| + \left|(RL)\int_S g_{k(\varepsilon)} d\mu - u\right| \leq \varepsilon.$$

□

Remark 3. *We can extend Proposition 1 to the bounded sequences $(g_n)_n$ that converge μ-almost uniformly on S (namely to the sequences $(g_n)_n$ such that for every $\varepsilon > 0$ there exists $B(\varepsilon) \in \mathcal{A}$ with $\mu(B(\varepsilon)) \leq \varepsilon$ and g_n converges uniformly to g on $S \setminus B(\varepsilon)$), if we assume that even g is bounded.*

We can proceed in fact in the same way, as in the previous proof, taking $P_\varepsilon^* := P_{\varepsilon/3,k(\varepsilon)} \wedge \{S \setminus B(\varepsilon), B(\varepsilon)\}$ and, for every countable partition $P := (A_n)_n$ finer than P_ε^*, dividing $\sum_{n=0}^{+\infty} g(t_n)\mu(A_n)$ in two parts: the one relative to $S \setminus B(\varepsilon)$, where the uniform convergence is assumed, and the remining part.

Convergence results in Gould integrability of functions with respect to a submeasure of finite variation are established for instance in [50].

Given two submeasures $\mu_1, \mu_2 : \mathcal{A} \to \mathbb{R}_0^+$ with $\mu_1(A) \leq \mu_2(A)$ for every $A \in \mathcal{A}$ let $M : \mathcal{A} \to ck(\mathbb{R}_0^+)$ defined by

$$M(A) = [\mu 1(A), \mu 2(A)]. \quad (1)$$

M is called an *interval submeasure*. For results in this subject see for example [3,43].

Let $M : \mathcal{A} \to ck(\mathbb{R}_0^+)$. We say that M is an *interval valued multisubmeasure* if

- $M(\emptyset) = \{0\}$;
- $M(A) \preceq M(B)$ for every $A, B \in \mathcal{A}$ with $A \subseteq B$ (monotonicity);
- $M(A \cup B) \preceq M(A) + M(B)$ for every disjoint sets $A, B \in \mathcal{A}$ (subadditivity).

In literature the multimeasures that satisfy the first two statements are also called set valued fuzzy measures (see for example [4] (Definition 1), [3,11,42–44] and the references therein).

A very interesting case of interval-valued multisubmeasure was given, for the first time, in [6,8] where Dempster and Shefer proposed a mathematical theory of evidence using non additive measures: Belief and Plausibility in such a way for every set A the *Belief interval* of the set is $[Bel(A), Pl(A)]$. This theory is capable of deriving probabilities for a collection of hypotheses and it allows the system inferencing with the imprecision and uncertainty. If the target space is $ck([0, 1])$ it is used for example in decision theory.

We say that M is an *additive multimeasure* if $M(A \cup B) = M(A) + M(B)$ for every disjoint sets A, $B \in \mathcal{A}$.

If a multimeasure M is countably additive in the Hausdorff metric d_H, then it is called a d_H-*multimeasure*. In this case we have that $\lim_{n\to\infty} d_H\left(\sum_{k=1}^{n} M(A_k), M(A)\right) = 0$, for every sequence of pairwise disjoint sets $(A_n)_n \subset \mathcal{A}$ such that $\cup_n A_n = A$.

Remark 4. *By Ref. [43] (Remark 3.6) $M(A) = [\mu_1(A), \mu_2(A)]$ is a multisubmeasure with respect to if and only if μ_1, μ_2 are submeasures in the sense of Definition 2 (iii). Moreover M is monotone, finitely additive, order-continuous, exhaustive respectively if and only if the set functions μ_1 and μ_2 are the same (see [40] (Proposition 2.5, Remark 3.3)).*

Definition 5. *Let $M : \mathcal{A} \to ck(\mathbb{R}_0^+)$. The variation of M is the set function $\overline{M} : \mathcal{P}(S) \to [0, +\infty]$ defined by*

$$\overline{M}(E) = \sup\{\sum_{i=1}^{n} \|M(A_i)\|, \{A_i\}_{i=1}^{n} \subset \mathcal{A}, A_i \subseteq E, A_i \cap A_j = \emptyset, i \neq j\}.$$

M is said to be of finite variation if $\overline{M}(S) < \infty$.

Remark 5. *We can observe that if $E \in \mathcal{A}$, then in the definition of $\overline{M}$ one may consider the supremum over all finite partitions $\{A_i\}_{i=1}^{n} \in \mathcal{P}_E$. If M is finitely additive, then $\overline{M}(A) = M(A)$, for every $A \in \mathcal{A}$.*

If M is subadditive (countably subadditive, respectively) of finite variation, then $\overline{M}$ is finitely additive (countably additive, respectively). Finally, if $M(A) = [\mu_1(A), \mu_2(A)]$, for every $A \in \mathcal{A}$, then $\overline{M} = \mu_2$.

3. *RL* Interval Valued Integral and Its Properties

In this section, we introduce and study Riemnn-Lebesgue integrability of interval-valued multifunctions with respect to interval-valued set multifunctions, pointing out various properties of this integral. For this, unless stated otherwise, in what follows suppose S is a nonempty set, with card $S \geq \aleph_0$ (card S is the cardinality of S), A is a σ-algebra of subsets of S.

The multisubmeasure M here considered is an interval-valued one and satisfies (1).

Given $g_1, g_2 : S \to \mathbb{R}_0^+$ with $g_1(s) \leq g_2(s)$ for all $s \in S$, let $G : S \to ck(\mathbb{R}_0^+)$ be the interval-valued multifunction defined by $G(s) = [g_1(s), g_2(s)]$ for every $s \in S$. For every countable tagged partition $\Pi := \{(B_n, s_n), n \in \mathbb{N}\}$ of S we denote by

$$\sigma_{G,M}(\Pi) := \sum_{n=1}^{\infty} G(s_n) \cdot M(B_n) = \sum_{n=1}^{\infty} [g_1(s_n)\mu_1(B_n), g_2(s_n)\mu_2(B_n)] =$$

$$= \{\sum_{n=1}^{\infty} y_n, y_n \in [g_1(s_n)\mu_1(B_n), g_2(s_n)\mu_2(B_n)], n \in \mathbb{N}\}.$$

By [20] (Lemma 2.2) the set $\sigma_{G,M}(\Pi)$ is closed and convex in $\mathbb{R}_0^+$, so it is an interval $[u(\sigma_{G,M}^{\Pi}), v(\sigma_{G,M}^{\Pi})]$.

Definition 6. *A multifunction $G : S \to ck(\mathbb{R}_0^+)$ is called Riemann-Lebesgue RL integrable with respect to M (on S) if there exists $[a, b] \in ck(\mathbb{R}_0^+)$ such that for every $\varepsilon > 0$, there exists a countable partition P_ε of S, so that for every tagged partition $P = \{(A_n, t_n)\}_{n\in\mathbb{N}}$ of S with $P \geq P_\varepsilon$, the series $\sigma_{G,M}(P)$ is convergent and*

$$d_H(\sigma_{G,M}(P), [a, b]) < \varepsilon. \tag{2}$$

$[a, b]$ is called the Riemann-Lebesgue integral of G with respect to M and it is denoted

$$[a, b] = (RL) \int_S G \, dM.$$

Obviously, if it exists, is unique.

Example 1. *Suppose $S = \{s_n | n \in \mathbb{N}\}$ is countable, $\{s_n\} \in \mathcal{A}$, for every $n \in \mathbb{N}$, and let $G : S \to ck(\mathbb{R}_0^+)$ be such that the series $\sum_{n=0}^{\infty} g_i(s_n)\mu_i(\{t_n\})$, $i = 1, 2$ are convergent. Then G is RL integrable with respect to M and*

$$(RL) \int_S G \, dM = \left[\sum_{n=0}^{\infty} g_1(s_n)\mu_1(\{s_n\}), \sum_{n=0}^{\infty} g_2(s_n)\mu_2(\{s_n\})\right]$$

Observe moreover that, in this case, the RL-integrability of such G with respect to M implies that the product G · G, as defined in **i***), is integrable in the same sense. In particular, if such G is a discrete or countable interval-valued signal, the* $(RL)\int_S G \cdot G\, dM$ *represents the energy of the signal.*

If M is of bounded variation and $G : S \to ck(\mathbb{R}_0^+)$ is bounded and such that $G = \{0\}$ M-a.e., then, by [34] (Theorem 3.4), G is M-integrable and $(RL)\int_S GdM = \{0\}$.

From now on we suppose that G is bounded and μ_2 is of finite variation.

Proposition 2. *An interval multifunction* $G = [g_1, g_2]$ *is RL integrable with respect to M on S if and only if* g_i *are RL integrable with respect to* μ_i, $i = 1, 2$ *and*

$$\int_S GdM = \left[(RL)\int_S g_1 d\mu_1,\ (RL)\int_S g_2 d\mu_2\right] \quad (3)$$

Proof. Suppose that $G = [g_1, g_2]$ is RL integrable with respect to $M = [\mu_1, \mu_2]$, that means there exists $[a, b] \in ck(\mathbb{R}_0^+)$ such that for every $\varepsilon > 0$, there exists a countable partition P_ε of S, so that for every tagged partition $P = \{(A_n, t_n)\}_{n\in\mathbb{N}}$ of S with $P \geq P_\varepsilon$, the series $\sigma_{G,M}(P)$ is convergent and

$$d_H([u(GP,M), v(GP,M)], [a, b]) := \max\{|u(GP,M) - a|, |v(GP,M) - b|\} < \varepsilon.$$

By this inequality it follows that

$$\max\{|\sum_{n=1}^{\infty} g_1(t_n)\mu_1(A_n) - a|, |\sum_{n=1}^{\infty} g_2(t_n)\mu_2(A_n) - b|\} \leq \varepsilon, \quad \forall\, n \in \mathbb{N},$$

for every tagged partition $P = \{(A_n, t_n)\}_{n\in\mathbb{N}}$ of S with $P \geq P_\varepsilon$ and then g_i are RL integrable with respect to μ_i, $i = 1, 2$. Formula (3) follows from the convexity of the RL integral.

For the converse, for every $\varepsilon > 0$, let P_{ε,g_i}, $i = 1, 2$ two countable partitions that verify the definition of RL integrability for g_i, $i = 1, 2$. Let P_ε be a countable partition of S with $P_\varepsilon \geq P_{\varepsilon,g_1} \wedge P_{\varepsilon,g_2}$. Then, for every $P := \{B_n, n \in \mathbb{N}\} \geq P_\varepsilon$ and for every $t_n \in B_n$ it is

$$\left|\sum_{n=0}^{+\infty} g_i(t_n)\mu_i(B_n) - (RL)\int_S g_i d\mu_i\right| < \varepsilon, \quad i = 1, 2.$$

Since g_i, $i = 1, 2$ are selections of G this means that

$$dH\left([u^{(P)}_{G,M}, v^{(P)}_{G,M}], \left[(RL)\int_S g_1 d\mu_1, (RL)\int_S g_2 d\mu_2\right]\right) \leq \varepsilon$$

and then the assertion follows. □

Remark 6. *By Definition 6 and Proposition 2 we obtain the following definitions for the following cases:*

- *If* $M = \{\mu\} : \mathcal{A} \to \mathbb{R}^+_0$ *is an arbitrary set function and* $G = [g_1, g_2]$ *with* $g_1(s) \leq g_2(s)$ *for every* $s \in S$ *then*

$$\int_S G dM = \left[(RL)\int_S g_1 d\mu, (RL)\int_S g_2 d\mu\right].$$

- *If* $M = [\mu_1, \mu_2]$ *as in (1) and* $G = \{g\} : S \to \mathbb{R}^+_0$ *then*

$$\int_S G dM = \left[(RL)\int_S g d\mu_1, (RL)\int_S g d\mu_2\right].$$

Proposition 3. *Let G be an interval valued multifuncion. The RL integrability with respect to M is hereditary on subsets* $A \in \mathcal{A}$ *. Moreover G is RL integrable with respect to M on A if and only if* $G\chi_A$ *(where* χ_A *is the characteristic function of the set A) is RL integrable with respect to M on S. In this case, for every* $A \in \mathcal{A}$*,*

$$(RL)\int_A G\, dM = (RL)\int_S G\chi_A\, dM.$$

Proof. Assume that G is RL integrable in S with respect to M. Let $A \in \mathcal{A}$ and denote by $[a, b]$ the integral of G; then, for every $\varepsilon > 0$, there exists a countable partition P_ε of S, such that, for every finer countable partition $P^0 := \{A_n\}_{n\in\mathbb{N}}$ and for every $t_n \in A_n$ it is

$$dH\left(\sigma_{G,M}(P'), [a, b]\right) \leq \varepsilon.$$

Let P_0 be a partition such that $P_0 \geq P_\varepsilon \wedge \{A, T \setminus A\}$, and we denote by $P_A \subset P_0$ the corresponding partition of the set A. Let Π_A be a partition of A finer than P_A, and extend it with a common partition of $S \setminus A$ in such a way the new partition is finer than P_ε.

It is possible to prove that $\sigma_{G,M}(\Pi_A)$ satisfy a Cauchy principle in $ck(\mathbb{R}^+_0)$, and so the first claim follows by the completeness of the space. The equality follows from [34] (Theorem 3.2) and Proposition 2. □

Remark 7. *It is easy to see that, if G is RL integrable with respect to M, for every* $\alpha \geq 0$ *it is:*

(a) αG *is RL integrable with respect to M and* $(RL)\int_S \alpha G\,dM = \alpha (RL)\int_S G\,dM$.

(b) G *is RL integrable with respect to* αM *and* $(RL)\int_S G\,d(\alpha M) = \alpha(RL)\int_S G\,dM$.

Theorem 1. *If G is an interval valued RL integrable with respect to M multifunction, then* $I_G : \mathcal{A} \to ck(\mathbb{R}_0^+)$ *defined by*

$$I_G(A) := (RL)\int_A G\,dM$$

is a finitely additive multimeasure.

Proof. By Proposition 3 we have that $I_G(A) \in ck(\mathbb{R}_0^+)$ for every $A \in \mathcal{A}$. In order to prove the additivity we can observe that, for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$

$$I_G(A \cup B) = (RL)\int_S G\chi_{A\cup B}\,dM = (RL)\int_S (G\chi_A + G\chi_B)\,dM. \quad (4)$$

If we prove that for every pair of interval valued RL integrable with respect to M multifunctions G_1, G_2 we have that

$$(RL)\int_S (G_1 + G_2)\,dM = (RL)\int_S G_1\,dM + (RL)\int_S G_2\,dM \quad (5)$$

the assertion follows. In order to prove formula (5) let $\varepsilon > 0$ be fixed. Since G_1, G_2 are RL integrable with respect to M, for every $\varepsilon > 0$ there exists a countable partition $P_\varepsilon \in \mathcal{P}$ such that for every $P = \{A_n\}_{n\in\mathbb{N}} \geq P_\varepsilon$ and every $t_n \in A_n$, $n \in \mathbb{N}$, the series $\sigma_{G_i,M}(P)$, $i = 1, 2$ are convergent and

$$d_H\left(\sigma_{G_i,M}(P), (RL)\int_S G_i\,dM\right) < \frac{\varepsilon}{2}, \qquad i = 1, 2.$$

Then $\sigma_{G_1+G_2,M}(P)$ is convergent and, by [48] (Proposition 1.17),

$$d_H\left(\sigma_{G_1+G_2,M}(P), (RL)\int_S G_1\,dM + (RL)\int_S G_2\,dM\right) < \varepsilon.$$

So $G_1 + G_2$ is RL integrable with respect to M and formula (5) is satisfied.

Now applying formula (5) with $G_1 = G\chi_A$, $G_2 = G\chi_B$ to formula (4) we obtain the additivity of I_G.

□

The set-valued integral is monotone relative to the order relation "" and the inclusion one, with respect to the interval-valued integrands.

Proposition 4. *If F, G are two RL integrable with respect to M interval valued multifunctions with F G then, for every* $A \in \mathcal{A}$, $I_F(A) \preceq I_G(A)$.

Proof. We will prove for $A = S$. Let $F(s) := [f_1(s), f_2(s)]$, $G(s) = [g_1(s), g_2(s)]$. By the integrability of F and G we have, by Proposition 2

$$I_F(S) := (RL)\int_S F\,dM = \left[(RL)\int_S f_1 d\mu_1, (RL)\int_S f_2 d\mu_2\right]$$

$$I_G(S) := (RL)\int_S G\,dM = \left[(RL)\int_S g_1 d\mu_1, (RL)\int_S g_2 d\mu_2\right]$$

Since $f_i(s) \le g_i(s)$ for all $s \in S$ and $i = 1, 2$ by [34] (Theorem 3.10) we have that

$$(RL)\int_S f_1 d\mu 1 \le (RL)\int_S g_1 d\mu 1, \qquad (RL)\int_S f_2 d\mu 2 \le (RL)\int_S g_2 d\mu 2,$$

and so by the weak interval order, **iii**), we have that $I_F(S) \preceq I_G(S)$. □

Corollary 1. *If* $F, G, F \wedge G, F \vee G$ *are RL integrable with respect to an interval valued multisubmeasure M then, for every* $A \in \mathcal{A}$,

(a) $(RL)\int_S F \wedge G\,dM \preceq I_F(A) \wedge I_G(A)$;

(b) $I_F(A) \vee I_G(A) \preceq (RL)\int_S F \vee G\,dM$.

Proof. Let $F(s) = [f_1(s), f_2(s)]$, $G(s) = [g_1(s), g_2(s)]$, $h_*(s) = \min\{f_1(s), g_1(s)\}$, $h^*(s) = \min\{f_2(s), g_2(s)\}$. By [34] (Theorem 3.10) $(RL)\int_S h_* d\mu_1 \le \left\{(RL)\int_S f_1 d\mu_1, (RL)\int_S g_1 d\mu_1\right\}$ and an analogous result holds for $(RL)\int_S h^* d\mu_2$. So the result given in 1.a) follows from the definition of $\preceq$ and $\wedge$.

The second statement follows analogously. □

Proposition 5. *Let* $F, G : S \to ck(\mathbb{R}_0^+)$ *be bounded so that F, G are RL integrable with respect to M. If* $F \subseteq G$, *then* $I_F(A) \subseteq I_G(A)$ *for all* $A \in \mathcal{A}$.

Proof. As before we will prove for S. Let $\varepsilon > 0$ be arbitrary. Since F, G are RL integrable with respect to M, there exists a countable partition Π_ε of S so that for every other countable partition $\Pi = \{B_n\}_{n\in\mathbb{N}} \in \mathcal{P}$, with $\Pi \ge \Pi_\varepsilon$ and every choise of points $s_n \in B_n$, $n \in \mathbb{N}$, the series

$$\sum_{n=0}^{\infty} F(s_n)\cdot M(B_n), \quad \sum_{n=0}^{\infty} G(s_n)\cdot M(B_n)$$

are convergent and

$$d_H\left(I_F(S), \sum_{n=0}^{\infty} F(s_n)\cdot M(B_n)\right) < \frac{\varepsilon}{3}; \quad d_H\left(I_G(S), \sum_{n=0}^{\infty} G(s_n)\cdot M(B_n)\right) < \frac{\varepsilon}{3}.$$

Then, by the triangular property of the eccess e,

$$\begin{aligned} e(I_F(S), I_G(S)) &\le d_H\left(I_F(S), \sum_{n=0}^{\infty} F(s_n)\cdot M(B_n)\right) + e\left(\sum_{n=0}^{\infty} F(s_n)\cdot M(B_n), \sum_{n=0}^{\infty} G(s_n)\cdot M(B_n)\right) + \\ &+ d_H\left(\sum_{n=0}^{\infty} G(s_n)\cdot M(B_n), I_G(S)\right) < \frac{2\varepsilon}{3} + e\left(\sum_{n=0}^{\infty} F(s_n)\cdot M(B_n), \sum_{n=0}^{\infty} G(sn)\cdot M(Bn)\right). \end{aligned}$$

Since the series $\sum_{n=0}^{\infty} F(s_n)\cdot M(B_n)$ and $\sum_{n=0}^{\infty} G(s_n)\cdot M(B_n)$ are convergent in $ck(\mathbb{R}_0^+)$, and, by hypothesis,

$\sum_{n=0}^{\infty} F(s_n)\cdot M(B_n) \subseteq \sum_{n=0}^{\infty} G(s_n)\cdot M(B_n)$, then

$$e\left(\sum_{n=0}^{\infty} F(s_n)\cdot M(B_n), \sum_{n=0}^{\infty} G(s_n)\cdot M(B_n)\right) = 0.$$

Consequently, from the arbitrariety of $\varepsilon > 0$, $e(I_F(S), I_G(S)) = 0$, which implies $I_F(S) \subseteq I_G(S)$. □

We can observe moreover that

Proposition 6. *If G is bounded and RL integrable with respect to M, with M of bounded variation, then*

(a) $\|I_G(S)\| = (RL)\int_S g_2\, d\mu 2 = (RL)\int_S \|G\|\, d\|M\|$.

(b)

$$\begin{aligned} \overline{I}_G(S) &= \sup\{\sum_{i=1}^{n} |I_G(A_i)|, \{A_i, i = 1, \ldots, n\} \in P\} = \\ &= \sup\{\sum_{i=1}^{n} (RL)\int_{A_i} g_2\, d\mu_2, \{A_i, i = 1, \ldots, n\} \in P\} = (RL)\int_S g2\, d\mu 2. \end{aligned}$$

Proof. It is a consequence of the properties of d_H and [34] (Proposition 3.3, Theorem 3.5). □

Proposition 7. *Let* $G : S \to ck(\mathbb{R}_0^+)$ *be a bounded multifunction such that G is RL integrable with respect to M on every set* $A \in \mathcal{A}$.

(a) *If M is of bounded variation, then* I_G $\overline{M}$ *(in the ε-δ sense) and* I_G *is of finite variation.*

(b) *If moreover M is o-continuous (exhaustive respectively), then* I_G *is also o-continuous (exhaustive respectively).*

Proof. The statements easily follow by Proposition 6. □

Moreover

Theorem 2. *Let* $G : S \to ck(\mathbb{R}_0^+)$ *be a multifunction such that G is RL integrable with respect to M on every set* $A \in \mathcal{A}$. *The following statements hold:*

(a) *If M is monotone, then* I_G *is monotone too.*

(b) *If M is a* d_H*-multimeasure of bounded variation then* I_G *is countably additive.*

Proof. Let $A, B \in \mathcal{A}$ with $A \subseteq B$. By monotonicity $\mu_i(A) \leq \mu_i(B)$ for $i = 1, 2$. We divide B in A, $B \setminus A$ and we apply [34] (Theorem 3.2, Corollary 3.6). The conclusion follows by (**iii**).

Since M is a d_H-multimeasure, then $\overline{M}$ is countably additive too and o-continuous. Applying Proposition 7 I_G is o-continuous too. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ be an arbitrary sequence of pairwise disjoint sets, with $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{A}$. We denote by B_n the set $B_n := \bigcup_{k=n+1}^{\infty} A_k$. Since $B_n \searrow \emptyset$, then $\lim_{n\to\infty} \|I_G(B_n)\| = 0$. Since I_G is finitely additive, we have

$$\lim_{n\to\infty} d_H\Big(I_G(A), \sum_{k=1}^{n} I_G(A_k)\Big) = \lim_{n\to\infty} d_H\Big(\sum_{k=1}^{n} I_G(A_k) + I_G(B_n), \sum_{k=1}^{n} I_G(A_k)\Big) \leq \lim_{n\to\infty}\|I_G(B_n)\| = 0$$

which ensures that I_G is a d_H-multimeasure. □

Proceeding as in to the proof of the formula (5) and applying [34] (Theorem 3.8) we obtain the following result:

Proposition 8. *Let be* $M_1, M_2 : \mathcal{A} \to ck(\mathbb{R}_0^+)$, *with* $M_1(\emptyset) = M_2(\emptyset) = \{0\}$ *and suppose* $G : S \to ck(\mathbb{R}_0^+)$ *is RL integrable with respect to both* M_1 *and* M_2. *If* $M : \mathcal{A} \to ck(\mathbb{R}_0^+)$ *is the interval-valued multisubmeasure defined by* $M(A) = M_1(A) + M_2(A)$, *for every* $A \in \mathcal{A}$, *then G is RL integrable with respect to M and*

$$(RL)\int_S G\, d(M_1+M_2) = (RL)\int_S G\, dM_1 + (RL)\int_S G\, dM_2.$$

Theorem 3. *Let M be of bounded variation and $F, G : T \to ck(\mathbb{R}_0^+)$ be bounded interval-valued multifunctions. If F, G are RL integrable with respect to M, then*

$$d_H\left((RL)\int_S F\,dM,\ (RL)\int_S G\,dM\right) \le \sup_{s\in S} d_H(F(s),G(s))\cdot \overline{M}(S).$$

Proof. Since F, G are M-integrable then f_1, g_1 are μ_1-integrable and f_2, g_2 are μ_2-integrable functions. According to [34] (Theorem 3.9), we have for $i = 1, 2$,

$$\left|(RL)\int_S f_i\,d\mu_i - (RL)\int_S g_i\,d\mu_i\right| \le \sup_{s\in S}|f_i(s)-g_i(s)|\,\overline{\mu}_i(S). \qquad (6)$$

Therefore, by (6) and Remark 5, it follows

$$d_H\left((RL)\int_S F\,dM,\ (RL)\int_S G\,dM\right) = \max\left\{\left|(RL)\int_S f_1 d\mu_1 - (RL)\int_S g_1 d\mu_1\right|, \left|(RL)\int_S f_2 d\mu_2 - (RL)\int_S g_2 d\mu_2\right|\right\}$$

$$\le \max\left\{\sup_{s\in S}|f_1(s)-g_1(s)|\,\overline{\mu_1}(S), \sup_{s\in S}|f_2(s)-g_2(s)|\,\overline{\mu_2}(S)\right\} \le$$

$$\le \max\left\{\sup_{s\in S}|f_1(s)-g_1(s)|, \sup_{s\in S}|f_2(s)-g_2(s)|\right\}\overline{\mu}_2(S) \le$$

$$= \sup_{s\in S} d_H(F(s), G(s))\,\overline{M}(S).$$

□

Theorem 4. *Let $M_1, M_2 : \mathcal{A} \to ck(\mathbb{R}_0^+)$ and $G : S \to ck(\mathbb{R}_0^+)$ be RL integrable with respect to both M_1 and M_2. Then*

(a) *If $M_1 \preceq M_2$, then $(RL)\int_S G\,dM_1 \preceq (RL)\int_S G\,dM_2$.*

(b) *If $M_1 \subseteq M_2$, then $(RL)\int_S G\,dM_1 \subseteq (RL)\int_S G\,dM_2$.*

Proof. Let $M_1 := [\mu_*, \mu^*]$ and $M_2 := [\nu_*, \nu^*]$. Both the results are consequences of Theorem 2 and [34] (Theorem 3.11). It is enough to observe that if $M_1 \preceq M_2$ then $\mu_* \leq \nu_*$ and $\mu^* \leq \nu^*$, while if $M_1 \subseteq M_2$ then $\nu_* \leq \mu_* \leq \mu^* \leq \nu^*$. □

As a particular case of Theorem 4 and Corollary 1 we have that for every G which is RL integrable with respect to both positive submeasures μ_1 and μ_2 then

$$(RL)\int_S G\, d(\mu_1 \wedge \mu_2) \preceq (RL)\int_S G\, d\mu_1 \wedge (RL)\int_S G\, d\mu_2.$$

Moreover a convergence result can be obtained using Proposition 1.

Theorem 5. *Let $G_n = [g_1^{(n)}, g_2^{(n)}]$ be a sequence of bounded RL-integrable interval valued multifunction with respect to $M = [\mu_1, \mu_2]$ such that $G_n \preceq G_{n+1}$ for every $n \in \mathbb{N}$. If M is of bounded variation and there exists a function $G = [g_1, g_2]$ such that:*

(a) $d_H(G_n, G) \to 0$ *uniformly;*

(b) $\sup_n \left\| (RL)\int_S G_n\, dM \right\| < +\infty$,

then G is RL-integrable with respect to M and

$$\lim_{n\to\infty} d_H\left((RL)\int_S G_n\, dM, (RL)\int_S G\, dM\right) = 0.$$

Proof. Since $G_n \preceq G_{n+1}$ we have that $g_i^{(n)} \uparrow$ for $i = 1, 2$, this is a consequence of Proposition 4 and Definition 6. By $d_H(G_n, G) \to 0$ uniformly we have that $\max\{|g_i^{(n)} - g_i|, i = 1,2\}$ converges uniformly
to zero. We can use now Proposition 1 and we obtain

$$\lim_{n\to\infty}(RL)\int_S g_i^{(n)}\, d\mu_i = (RL)\int_S g_i\, d\mu_i, \quad i = 1, 2.$$

For every $\varepsilon > 0$ let $k(\varepsilon) \in \mathbb{N}$ be such that $d_H(G(t), G_{k(\varepsilon)}(t)) < \varepsilon \quad \forall\ t \in S$, and $\left|(RL)\int_S g_i^{(k(\varepsilon))}\, d\mu_i - (RL)\int_S g_i\, d\mu_i\right| < \varepsilon,\ i = 1, 2.$

So, $d_H\left((RL)\int_S G_{k(\varepsilon)}\, dM, \left[(RL)\int_S g_1 d\mu_1, (RL)\int_S g_2 d\mu_2\right]\right) \leq \varepsilon$.

Let P_ε be the countable partition of S given by $\vee_{i=1,2} P_{\varepsilon,i}$, (the ones that verify Definition 4 for $g_i^{k(\varepsilon)}$, $i = 1, 2$ respectively). Then, for every countable partition $P = \{A_n\}_{n\in\mathbb{N}}$ of S with $P \geq P_\varepsilon$ and for every $t_n \in A_n$ the series $\sigma_{G,M}(P)$ is convergent and

$$d_H\left(\sigma_{G,M}(P), \left[(RL)\int_S g_1 d\mu_1, (RL)\int_S g_2 d\mu_2\right]\right) < d_H\left(\sigma_{G,M}(P), \sigma_{G_{k(\varepsilon)},M}(P)\right) +$$
$$+ d_H\left(\sigma_{G_{k(\varepsilon)},M}(P), (RL)\int_S G_{k(\varepsilon)} dM\right) +$$
$$+ d_H\left((RL)\int_S G_{k(\varepsilon)} dM, \left[(RL)\int_S g_1 d\mu_1, (RL)\int_S g_2 d\mu_2\right]\right).$$

From previous inequalities and by the arbitrariety of ε the RL-integrability of G follows. □

Remark 8. *Since this research starts from the papers [34,43], this part ends with a comparison between the two types of integral considered: the RL integral with the Gould one given in [43] (Definition 4.7).*

If the interval-valued multifunction F is bounded and μ_2 is of finite variation then, analogously to Proposition 2 it is, by [43] (Proposition 4.9),

$$(G)\int_S F dM = \left[(G)\int_S f_1 d\mu_1, (G)\int_S f_2 d\mu_2\right]$$

So, the two kinds of integral coincide on bounded interval-valued multifunctions with values in $ck(\mathbb{R}_0^+)$ when μ_i, $i = 1, 2$ are complete countably additive measures by [34] (Proposition 4.5) or μ_i, $i = 1, 2$ are monotone, countably -subadditive by [34] (Theorem 4.7).

Without countable additivity the equivalence does not hold; an example can be constructed using [34] (Example 4.6). In the general case only partial results can be obtained on atoms when μ_i, $i = 1, 2$ are monotone, null additive and satisfy property (σ): the proof follows from [34] (Theorem 4.8).
Accordingly with the comparison between Gould and Birkoff integrals given in [28] we have that Birkhoff,
Gould, RL integrals of the bounded single valued functions agree in the countably additive case, see [28] (Theorem 3.10), while in [43] (Remark 5.5) an analogous comparison is given with the Choquet integral.

A comparison between simple Birkhoff and RL integrabilities, introduced in [23,28], in this non additive setting can be obtained using [34] (Theorem 4.2).

Finally we would like to observe that the Rådström's embedding tell us that $(ck(X), d_H, \subseteq)$, when X finite dimensional, is a near vector space with 0 element and order unit B_X. In this case, using [51] (Theorem 5.1), it is a near vector space (see [51] (Definition 2.1) for its definition) that could be embedded, for example, in ℓ^∞ or in $C(\Omega)$ with Ω compact and Hausdorff in such a way the embedding is an isometric isomorphism which takes into account the ordering on the hyperspace.

If we consider instead $(ck(\mathbb{R}_0^+), d_H, \preceq)$, since in general there is no relation between "" and "⊆" the Rådström embedding provide only the integrability of the interval−valued functions and does not take the weak interval order into account. For this reason we preferred to give the the construction of the *RL* integral and the proofs, both related to , independently of the Rådström's embedding.

3.1. Applications of Interval Valued Multifunctions

Now, in order to explain what could be the benefits of this approach we give an example of an application of interval valued multifunctions on interval valued multisubmeasure in image processing. In fact a signal can be modeled as an interval-valued multifunction as in [12]. In fact, when the value of the points can not be assigned with precision, it might be preferable to use a measure-based approach.

The advantage of using the notion of interval-valued multifunction in signal analysis is that this formalism allows to include in a unique framework possible uncertainty or the noise on the value of a point.

This situation usually occurs in signal and image processing when images are derived by a measure process, as happens for instance for biomedical images (in CT images, MR images, etc), and in several other applied sciences. In particular, we can apply this representation to a digital image in such a way:

Example 2. *To each pixel (or to a set of pixels) of the image is associated an interval which measures the* round-off error *which is that committed on the detection on the signal due by the tolerances and by the limits on computational accuracy of the measurements tools ([52]).*

When we consider subsets of pixels *we are taking into account the so-called* time-jitter error*, i.e., the error that occur in the measure of a given signal when the sampling values can not be matched exactly at the* theoretical node *but just in a neighborhood of it (see, e.g., [53]).*

In this sense, if $I = (m_{i,j})$ *is the matrix associated to a* $n \times m$ *static, gray-scale image, we can consider the space* $S := (0, n] \times (0, m] \subset \mathbb{R}^2$*, and hence the interval-valued multifunction* $U_I : \mathcal{S} \to \mathcal{K}_C^+$ *corresponding to I, will be given by:*

$$U_I(x) := [u_1(x), u_2(x)], \qquad x \in S.$$

The model of a digital image by an interval-valued multifunction as U_I*, and obtained by a certain discretization*
(algorithm) of an analogue image, allows to control the round-off error in the sense that, the true value assumed by original signal at the pixel x belongs to the interval $[u_1(x), u_2(x)]$*, in fact providing a lower and an upper bound on the possible oscillations of the sampled image.*

For example, in fractal image coding, the functions u_1 *and* u_2 *represent respectively the lower and upper contraction maps of an image, which take into account of the round-off error in the contraction procedure, and can be chosen as follows:*

$$u_1(x) := \alpha 1\ u(x) + \beta 1(x), \qquad u_2(x) := \alpha 2\ u(x) + \beta 2(x), \qquad x \in S,$$

where α_i*,* $i = 1, 2$*, are suitable integer scaling parameters,* $\beta_i : S \to \mathbb{N}$*,* $i = 1, 2$*, are suitable functions, and* $u : S \to \mathbb{N}$ *is the continuous model associated to the starting image I. The functions* u_1 *and* u_2 *provide for each pixel the interval containing the true value of the compressed image.*

In particular, in the algorithm considered in [15], the functions u_1 *and* u_2 *are piecewise constant, and for a starting image of* 225 × 225 *pixel size, they have been defined as follows:*

$$U_I(x) = [u_1(x), u_2(x)] = [u(x) - \beta(x), u(x) + \beta(x)], \qquad x \in S, \tag{7}$$

where:

$$u(x) := m_{i,j}, \quad x \in (i-1, i] \times (j-1, j], \; i, j = 1, ..., 225,$$

and

$$\beta(x) := \begin{cases} 0, & x \in (0, 115] \times (0, 115], \\ 40, & x \in (115, 225] \times (115, 225], \\ 20, & \text{otherwise.} \end{cases} \tag{8}$$

As an example we use the interval-valued multifunction (7) to operate with the well-known image of "Baboon" given in Figure 1 (left); the images generated by u_1 *and* u_2 *using the function* β *defined in (8) are given in Figure 1 (center and right).*

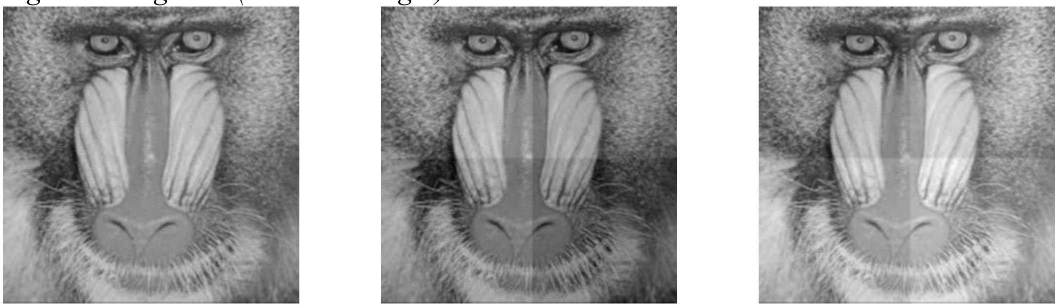

Figure 1. Baboon (**left**); The images generated by u_1 (**center**) and u_2 (**right**) using the interval valued multifunction (7), with β defined in (8).

Here, also numerical truncation have been taken into account, in order to maintain the values of the pixels in the (integer) gray scale [0, 255].

For other examples of functions u_1 *and* u_2*, see, e.g., [13,54]. For instance, in [13] the image representation by multifunctions is used for the implementation of edge detection algorithms, and in this case the corresponding functions* u_1 *and* u_2 *are:*

$$u_1(x) := \max\left\{0, \min_{x' \in n(x)} \{I(x') - 1\}\right\}, \qquad u_2(x) := \min\left\{255, \max_{x' \in n(x)} \{I(x') + 1\}\right\},$$

where $I(x)$ *represents the value of a pixel at a position* $x \in S$*, while* $n(x)$ *denotes any set of* 3 × 3 *pixels centered at x. For more details, or other applications, see [13,18].*

This example was built with the aim to highlight a useful link between the abstract theory of the interval-valued multifunction and the concrete application to image processing. One of the crucial tool in the above set-valued theory is provided by the Hausdorff distance between sets. This special metric plays an important role in the context of digital image processing, where it is used, for example, in order to measure the accuracy of certain class of algorithms, such as those of edge detection, already mentioned in the previous list of possible applications. More precisely, if A is the region of interest (ROI) of a given image and B is the corresponding approximation of the ROI A detected by a suitable edge detection algorithm, the Hausdorff distance measure the displacement between A and B, in fact evaluating the accuracy (i.e., the approximation error) of the method. For instance, in [55] the Hausdorff distance has been used in order to evaluate the degree of accuracy of an algorithm for the detection of the pervious area of the aorta artery from CT images without contrast medium. This procedure is useful, for example, in the diagnosis of aneurysms of the abdominal aorta artery, especially for patients with severe kidneys pathology for which CT images with contrast medium can not be performed. A similar use of the Hausdorff distance could be done for the edge detection algorithms considered in [13,18].

4. Conclusions

A Riemann Lebesgue integral is defined for interval-valued multifunction with respect to interval-valued multisubmeasures. Properties of the integral are established showing in particular that the multimeasure generated is finitely additive. Sufficient conditions for the monotonicity, the order continuity, bounded variation and convergence results are also obtained. A comparison with other integrals is sketchced; an example of an applications in image processing is given highlighting that the advantage of using the notion of interval-valued multifunction in signal analysis is that this formalism allows to include in a unique framework possible uncertainty or the noise on the evaluation of an image at any given pixel. In a future research we will generalize these results in the setting of Banach lattices and we will compare this method with other DIP (digital image processing) algorithms.

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