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Advancements in Fractional Calculus: Mixed Integrals within Weighted Hölder Spaces

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Abstract: *We consider operators of mixed fractional integration in weighted generalized Hölder spaces of a function of two variables defined by a mixed modulus of continuity.*

Keywords: *functions of two variables, mixed fractional integral, mixed difference, generalized Hölder spaces, weighted spaces, modulus, mixed modulus of continuity.*

1. Introduction

One of the most important problems in the theory of integral operators in space is the problem of elucidating the dependence of the smoothness of the image on the smoothness of the preimage. The solution to such a problem plays an important role in the solvability of integral equations, their stability, and so on. The concept of smoothness can be formulated in a variety of terms. One of the ways of sufficiently fine-grabbing the smoothness of functions is the notion of generalized Hölderness, formulated in terms of the behavior of the modulus of continuity.

Thus, one of the important questions in the theory of operators is as follows: Let be A an operator acting in a Banach space X and let be the modulus of continuity $\cdot\cdot f; h\cdot\cdot \sup_{|t|\cdot h} \|f\cdot x \cdot h\cdot\cdot f\cdot x\cdot\|_X$ of X. How can the behavior of the modulus of continuity be characterized $\cdot\cdot A\cdot, h\cdot$ if the behavior of the modulus of continuity of a function $\cdot\cdot\cdot; h\cdot: \cdot\cdot\cdot; h\cdot\cdot C\cdot\cdot h\cdot$ for all is known $\cdot\cdot X$, where is $\cdot\cdot x\cdot$ a given continuous function, $\cdot\cdot 0\cdot\cdot 0$.

A similar problem can be considered completely solved for different spaces, and also for the Hölder space of functions of one variable and power weights, when

$\cdot A\cdot\cdot\cdot\cdot x\cdot\cdot\cdot^{1}\cdot\cdot\cdot t\cdot\cdot^{1}*\cdot\cdot, 0 \cdots 1$ ([2] - [6], [8] – [13]). A detailed review of these and some other close results can be found in [10]. The assertion for multidimensional cases on the property of mapping in the usual Hölder and in the Hölder spaces defined by mixed differences are known [7]. Also, f generalized Hölder space is known for the Riesz fractional integral [13] (see also [12], Theorem 25.5). Mixed fractional Riemann-Liouville integrals of order $\cdot\cdot, \cdot\cdot$

$$\cdot I\cdot_{0\cdot,0\cdot\cdot\cdot\cdot\cdot}, \cdot \cdot x, y\cdot\cdot \quad \frac{1}{\cdot\cdot\cdot\cdot\cdot\cdot\cdot} \int_0^x\int_0^y \frac{\cdot\cdot t,\cdot\cdot dtd\cdot}{\cdot x\cdot t\cdot^{1\cdot\cdot}\cdot y\cdots^{1\cdot\cdot}}, \qquad (1.1)$$

where $x, y \cdot 0,\cdot,\cdot\cdot0,1\cdot$ have not been studied.

This paper is devoted to the study of certain properties of the mixed fractional integral (1.1) in weighed generalized Hölder spaces of a function of two variables defined by a mixed modulus of continuity.

We consider the operator (1.1) in $Q \cdot\cdot\cdot x, y\cdot: 0 \cdot x \cdot b, 0 \cdot y \cdot d\cdot$.

2. Preliminary information and notations

When studying the properties of continuous functions of several variables, in particular, two variables, the following classes of functions arise:

$$H^{\cdot_1,\cdot_2,\cdot_{1,1}} \cdot \cdots \cdot\cdot x, y\cdot\cdot \mathbf{C}Q : \cdot^{1,0}\cdot\cdot;\cdot,0\cdot\cdot\ O\cdot\cdot_1\cdot\cdot\cdot,$$

$$\cdot^{0,1}\cdot\cdot;0,\cdot\cdot\ O\cdot\cdot_2\cdot\cdot\cdot,\ \cdot^{1,1}\cdot\cdot;\cdot,\cdot\cdot\ O\cdot\cdot_{1,1}\cdot\cdot,\cdot\cdot\cdot,\ \cdot\cdot,$$

$$H^{\cdot_1,\cdot_2}\cdot\cdot\cdot\cdot x, y\cdot\cdot \mathbf{C}Q :\cdot^{1,0}\cdot\cdot;\cdot,0\cdot\cdot\ O\cdot\cdot_1\cdot\cdot\cdot,\cdot^{0,1}\cdot\cdot;0,\cdot\cdot\ O\cdot\cdot_2\cdot\cdot\cdot,$$

where $\cdot^{1,0}\cdot\cdot;\cdot,0\cdot\cdot \sup \ \sup \left|\cdot^{1,0}_{h}\cdot\cdot\cdot x, y\cdot\right|$, $y\ h\cdot(0,\cdot]$ $\cdot$

$\cdot^{0,1}\cdot\cdot;0,\cdot\cdot \sup \ \sup \left|\cdot^{0,1}\cdot\cdot\cdot\cdot x, y\cdot\right|$ - are the partial modulus of $x\ \cdot\cdot(0,\cdot]$ continuity of the first order, and a

$\cdot^{1,1}\cdot\cdot;\cdot,\cdot\cdot \sup \sup \left|\cdot^{1,1}_{h,\cdot\cdot}\cdot x, y\cdot\right|$ is a mixed modulus of $x, y\ 0\cdot h\cdots\ 0\cdots$ continuity of order (1,1);

$$\cdot^{1,0}_{h}\cdot\cdot\cdot x, y\cdot\cdot\cdot x \cdot h, y\cdot\cdot\cdot x, y\cdot;$$

$$\cdot^{0,1}\cdot\cdot\cdot\cdot x, y\cdot\cdot\cdot x, y \cdot\cdot\cdot\cdot x, y\cdot;$$

$$\cdot^{1,1}_{h,\cdot}\cdot\cdot x, y\cdot\cdot\cdot x \cdot h, y \cdot\cdot\cdot\cdot x \cdot h, y\cdot\cdot\cdot x, y \cdot\cdot\cdot\cdot x, y\cdot,$$

$\cdot_1,\cdot_2\ \cdot\cdot^{1},\cdot_{1,1}\cdot\cdot^{1,1}$ (definition of classes $\cdot^{1}$ and $\cdot^{1,1}$ see below).

The following identity is valid

$$\cdot\cdot x \cdot h, y \cdot\cdot\cdot\cdot^{1,1}_{h,\cdot}\cdot\cdot x, y\cdot\cdot\cdot \cdot^{1,0}_{h}\cdot\cdot\cdot x, y\cdot \qquad (2.1)$$

$\cdot^{0,1}\ \cdot$

·· ·.··x, y···x, y·
· ·
· ·

Definition 2.1. Let function .·x· is a bounded on [a, b] . The modulus of continuity of··x· is the expression

··;··   sup··x1···x2·, x1, x2·[a, b] x1·x2 ··

is defined for all · that satisfy the condition 0 ···b·a .

Definition 2.2. A function ···· ·0 ···b·a· is called a modulus of continuity if it satisfies conditions

1) lim ···· 0 ;
··0
2) ··· is almost increasing on (0,b · a] ;
3) ···1 ··2····1····2·;
4) ··· is function continuous in · on (0,b · a]

Definition 2.3. We denote by .¹ the class of functions .·.· defined on (0,b · a] , and satisfying conditions a) ··· is a modulus of continuity

b) ··___·t dt · C··· ;
0 t

c) ·b· a·t·2t·dt · C···;
·

d) ·····~ ···· ·
·

It follows from the definition ··;·,·· that this function belongs to .¹ each of the variables. In addition, we note the inequality

1·,1··;·,·· 2min ···1·,0··;·,0·, 0·,1··;0,····. (2.2)
·· ··

Definition 2.4. We denote by . two variables ··,·· satisfying conditions: 1) ··,··in · for any fixed ·; 1,1·Q· the class of functions of
2) ··,··in · for any fixed ·.

We call this class the class of mixed modulus of continuity of the first order of continuous functions of two variables.

In [1] was shown that the properties 1) and 2) are characteristic for continuity modulus in the sense that for every ...1,1 there exist such a function ··CQ , that

1,1 1,0 0,1
···;·,··~1,1··,··, ···;·,0··1···, ···; 0,··~·2···.

Definition 2.5. Let us denote · the set of satisfying

··1,1,·1,·2·

1) ·1···,·2····.¹ ;
2) ·1,1··,···.1,1;
3) ·1,1··,·· Cmin··1···,·2····, where C - is not envy from ·1,·2,·1,1.

Let ·· ··1,·2,·1,1···. . .1..1..1,1 . We have introduced a norm in H~··def· H··1,·2,·1,1·space

‖ ‖ ·H~··H··1,·2,·1,1· · max C.1,0, C.0,1, C.1,1 ,

where

1,0 0,1
C·1,0 · sup______ C·0,1 · sup ______···;
···;·,0·, ··0 ·1··· 0,··, ··0 ·2···
1,1
C·1,1 · sup ______
···;·,··, ··0, ··0 1,1··· ‖C·Q· ··x,maxy··Q··x, y· .
~

Definition 2.6. We say that ··x, y··H 0 ·Q·, if .·x, y··H ~ ·Q· and ··x, y·x 0, y·0 ···x, y·x·b, y·d · 0 .

We will also make use of the following weighted spaces. Let ··x, y· be a non-negative function on Q (we will only deal with degenerate weights ··x, y····x···y·.

Definition 2.7. By H~ ·Q,··. H~ ··· we denote the space of ~, respectively, equipped functions ··x, y· such that ···H with the norm

‖ ‖ ·H~··· ··‖H‖.

~ ···· we denote the corresponding subspaces of By H0 functions ··x, y· such that

··x, y···x, y·x·0, y·0 ···x, y···x, y·x·b, y·d · 0 .

Below we follow some technical estimations suggested in [11] for the case of one-dimensional Riemann - Liouville fractional integrals. We denote

B·x, y;t,·· __________________··t,····x·x·, ty···1······yt,···· ·1·· ,

where 0 ··,··1; 0 · t · x · b, 0 ··· y · d . In the case ··x, y····x···y· we have

B·x, y; t,·· B1·x,t·B2·y,·· ·yB·1··x·,1t·· · ·xB2··ty·1,···· , (2.3)

where

$B_1•x,t•· ··\overline{11•t••x•x····t1}•1•·t·•$, $B_2•y,\overline{•· ··22•···y••y····}2•1•···•$.

Let also

$D_1•x,h,t•· B_1•x · h, t•· B_1•x,t•, t,x,x · h·[0,b], h · 0;$

$D_2•y,·, t•· B_2•y ··,·• B_2•y,·•, ·, y, y ···[0,d],·· 0$. **Lemma 2.1**. ([3]) Let $·1•x•· x^{·}, ··\mathbf{R}^1, 0 ···1$. Then

$$B_1•x, t• · C·^{·} x ··\max•··1,0• •x ·t•· , \quad (2.4)$$

$$| \quad | \quad ·\perp· \quad \overline{\quad} t$$

$$|D_1•x, h, t• |·C··· \overline{x·t\ h}···\max•··1,0•\overline{\qquad\qquad} t•x·hh·t•1·· . \quad (2.5)$$

Similar estimates hold for $B_2•y,·•$ and $D_2•y,·,·•$ with $·2•y•· y^{·}$.

Remark 2.1. All the weighted estimations of fractional integrals in the sequel are based on inequalities (2.4)-(2.5). Note that the right - hand sides of these inequalities have the exponent $\max•··1,0•$, which means that in the proof it suffices to consider only the case $··1$, evaluations of $··1$ being the same as for $··1$. The following statements are known, begin first proved in (see also [12], p. 197). However, here we give a sketch of the proof of this lemma, in order to compose the representation of lightness for the two-dimensional case. Consider the onedimensional fractional Riemann-Liouville integral

$$•I_{0·}^{·}•·x•· ·\overline{·^{1}}·_0^x· \overline{·x ·^{·}t ·^{t}•^{·}1} ·· dt, x · 0,0 ···1. \quad (2.6)$$

Theorem 2.1. Let $··x•$ be continuous on $[0, b]$ and let $··0•· 0$. For the fractional integral (2.6), the estimate

$$• \quad • \quad \overline{\qquad} \quad b•··, t•dt \quad (2.7)$$
$$·I·0··, h · Ch· t2·· $$
$$h$$

is valid.

Proof. Representing (2.6) as

$$•I·0··•·x•· ··•··0•· \overline{0x}· •x ·\overline{dtt•1··} · ··1·• \overline{0x}··\overline{•x•t·•·t•·1·•0•dt} · .$$
$$· A_1•x•· A_2•x•$$

Let $h · 0; x,x · h·[0, b]$. We have

$$A_2•x · h•· A_2•x•· ··•x•1•\underline{·····•0·^{•}•x} · h•· · x·^{•}·$$

$$· ·•1·^{•} \overline{h0}··\overline{•x · t•1····•t}•dt · h ·t•$$

$$·^{1} x. \quad • ··1 \quad t^{·1}•dt ··1 ··2 ··3 .$$
$$•··x ·t•···t•.•h ·t• \quad .$$

$$·•·•0$$

We have: $|·1 |·C·•·;x••x· h•^{·}|·| x|$. In the case $x · h$ we have $·1 ·Ch^{·}•·;h•$. Let $|x| h$. Then

$$|·1 · |C·•·;x•x·····1· \underline{\ hx} ····1···· C·\underline{\qquad\quad}•1··;·x•h . \quad (2.8)$$
$$·· \quad · \quad · \quad x$$

Since

$$Cx··1•·;x•···•·;x•bx•t··2dx•bx··2•···;t•\overline{dt ·b··t2}•···;t•\overline{dt . t \ \ h}$$

It follows from (2.8) that

$$|·1|· Ch^{b}··\overline{t_2•··;t}\ dt .$$
$$h$$

Further,

$$|·2 ·|h· ··~~-;1t•··1··-~~; h•· \quad \overline{\qquad\qquad} \quad d·· Ch··•·; h•, dt · h$$
$$0 •h·t• \quad\quad 0 •1··•1··$$

with $C · ··\int_0^1 1···^{·1}d·$. To estimate $·3$ we distinguish the case 1) $x · h$ and 2) $x · h$. In the first case

$$| ·3 |· C··h··(f,t)^{•}t^{··}1 ··h·t··1^{•}dt ·$$
$$··0$$

$$x \quad • \quad ^{1}•h ·t•^{··} • \quad .$$
$$^{·}··(f,t)\ t^{··} · 1\ dt·· h \quad ··$$

$$· C_2·· · b··f,t•dt·· . h \ ·\underline{f,\ h··} h·$$
$$·· \quad\quad h\ t2·· ··$$

Obviously in the second case $|·3| ·C1h^{·}··f;h•$.

Estimates for $·1, ·2, ·3$ the lead to (2.7) if we take into account the fact that $h^{·}·•·; h•$ is dominated by the right-hand side of (2.7). The latter is easily obtained in view of the monotonicity of the function $·•·; t•$.

To obtain estimates of the Zygmund type in the weighted case, we use the notation and the proof scheme from [2] and [6].

Theorem 2.2. Let $·(x) = x^{·}, 0··< 2··$. If the function $f(x), x·[0,b]$ satisfies the condition:

1) $·(x)f(x)·\mathbf{C}[0,b]$ and $·(x)f(x)|_{x=0}= 0$;

2) the integral $·\int_0^b \frac{·(f,t)}{t}. dt$ converges for $·= \max(1,·)$.

Then estimates of the Zygmund type

$\cdot \cdot\ 1h\cdot(\cdot f,t)\ dt \cdot hb\cdot\cdot(2\cdot\cdot f\cdot,t)\ dt\cdots.\ (2.9)\ (\cdot I\cdot 0\cdot f,h) \cdot C\cdot\cdot h\cdots\ 0\cdot\ t\cdot h\ t \cdot$

Proof. We denote this $g(x) = \cdot(x) f(x)$. We have

$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet\ 1 \quad x$

$(\cdot I'0\cdot f)(x) = I'0\cdot g\ (x) \cdot J'0\cdot g\ (x),\ J'0\cdot g\ (x) = \cdot(\cdot)\ 0\cdot B(x,t)g(t)dt.$

Here the estimates for $\bullet I\cdot 0\cdot g\bullet(x)$ are solved in Theorem 2.1.

Now consider the difference

$\bullet J'0\cdot g\bullet(x \cdot h) \cdot\bullet J'0\cdot g\bullet(x) = F1(x,h) \cdot F2(x,h)$, where

$F1(x,h) = \cdot \overset{x\cdot h}{\underset{a}{}} B(x \cdot h,t)g(t)dt, \qquad F2(x,h) = \cdot \overset{x}{} D(x,h,t)g(t)dt.\ x$

Taking into account Remark 1.1, we consider only the case $1\cdot\cdot<2\cdot\cdot$. From (2.4) we have

$| F1 |\cdot C \cdot \cdot \cdot \cdot (g,t)dt.\ x \cdot t \cdot t\quad x\cdot h\cdot\ x\cdot h\cdots 1\ (x\cdot h\cdot t)\cdot$

If $x \cdot h$, then

$| F1 |\cdot Ch\cdots 1x\cdot\cdot h\cdot(tg\cdot;t)\ dt\ .\ x$

Using the property of almost decreasing $\frac{\cdot(g;t)}{t}$, we obtain

$| F1 |\cdot Ch\cdots 1x\cdot x\cdot h\cdot \underline{\qquad}(g;t\ \cdot\cdot x)\ dt = Ch\cdots 1h\cdot\cdot(tg\cdot;t)\ dt.\quad (t \cdot x)\quad 0$

If $x > h$, then

$| F1 |\cdot Ch\cdot(x\cdot h)\cdot\cdot 1x\cdot x\cdot h\cdot(tg\cdot;t)\ dt = C\ (x\cdot hh\cdot)\cdot\cdot 1\ h0\cdot\cdot((xg\cdot;xt)\cdot\cdot t)\ dt \cdot$

$\cdot\ Ch\cdot x\cdot\cdot 10h\cdot\cdot\underline{\qquad}\ \underline{\quad}(x(g\cdot;tx)\cdots t1)\ xdt\cdot\ t \cdot Ch\cdot h0\cdot\cdot\underline{\qquad}(gx;\cdot xt\cdot\ t)\ dt.$

Further, it is clear that

$| F1 |\cdot Ch\cdots\underline{\quad}(g;t)\ dt.\quad \overset{h}{\underset{0}{}}\ t$

Collecting the estimates $F1$, we obtain the inequality for $0\cdot\cdot<2\cdot\cdot$

$| F1 |\cdot Ch\cdots 1\cdot\ \overset{h}{\underset{0}{}}\ \frac{\cdot(g;t)}{t\cdot}\ dt,\ \cdot = \max(1,\cdot)\ .$

We pass to the estimate $F2$. Using the estimate (2.5), we obtain

$| F2 |\cdot Ch0x\cdots x\cdot t\ h\ \cdots\cdot 1\ (x\cdot\cdot h(\cdot g;tt)1)\cdot\cdot t\ dt\ . \qquad (2.10)$

When $h\cdot x$,

$| F2 |\cdot Ch\cdots 10x\cdot\cdot\underline{\qquad}(g\cdot;t)\ dt \cdot Ch\cdots 1h\cdot\cdot\underline{\qquad}(tg\cdot;t)\ dt.\ t0$

If $h\cdot x$, then, we represent the right-hand side of (2.10) as a sum of three terms:

$F2\cdot = Chh0\cdots x\cdot t\ h\ \cdots\cdot 1\ (x\cdot\cdot h(g\cdot;tt))1\cdot\cdot dtt\ ,$

$F2\cdot\cdot = Ch\ \overset{\frac{1}{2}(x\cdot h)}{\underset{h\cdot}{\cdot}}\ \cdot\ \frac{\cdot x \cdot h \cdot}{t}\cdot^{\cdot\cdot 1}\ \frac{\cdot(g;t)}{(x \cdot h\ \cdot t)^{1\cdot\cdot}}\ \frac{dt}{t},$

$F2\cdots = Ch\overset{x}{\underset{\frac{1}{2}(x\cdot h)}{1}}\cdot\ \cdots x\cdot t\ h\ \cdots^{\cdot\cdot 1}\ (x\ \cdot\cdot h(g\cdot;tt))1\cdot\cdot\ dtt\ .$

Then $| F2 |\cdot F2\cdot \cdot F2\cdots F2\cdots$.

For the term $F2\cdot$ the relations are valid $x\cdot h\cdot\ 2(x\cdot h\cdot t)$, therefore $F2\cdot\ \cdot\ Chh0\cdot\ \underline{\qquad\qquad}t\cdot(x\cdot\cdot\ (hg\cdot;tt))dt\ \cdot$

$Ch\cdots 10h\cdot\cdot\underline{\qquad}(g;t)\ dt.\quad 2\cdots\quad t\cdot$

For the summand, $F2\cdot\cdot$ we have $2t \cdot x \cdot h$, so $1\cdot\cdot<2\cdot\cdot$ we obtain the estimate

$F2\cdots\ Ch\ \overset{\frac{1}{2}(x\cdot h)}{\underset{h\cdot}{\int}}\ \frac{\cdot(g;t)}{t\cdots\ x \cdot h\ \cdot\cdot 2\cdots}\ dt \cdot Chh\cdot \overset{b}{}\ \frac{\cdot(g;t)}{t2\cdot\cdot}\ dt.\quad \cdot\ 2\ \cdot$

We estimate the term $F2\cdots$. Here $t \cdot x \cdot h\cdot t$, therefore $\frac{\cdot(g;t)\ \cdot(g;x \cdot h\ \cdot t)}{\ }\ \cdot C$, it follows that $t \quad x \cdot h\ \cdot t$

$F2\cdots\ Ch\ \overset{x}{\underset{\frac{1}{2}(x\cdot h)}{\int}}\ \frac{\cdot(g;x \cdot h\ \cdot t)}{(x \cdot h\ \cdot t)2\cdot\cdot}\ dt.$

Because $x \cdot h \cdot 2t$. Having made the change $\cdot = x\cdot h\cdot t$ and going back to the variable t, we get

$F2\cdots\ Ch\cdot\overset{b}{}\ \frac{\cdot(g;t)}{2\cdot\cdot}\ dt.$

$h \; t$

From the estimates $F2\cdot, F2\cdot\cdot, F2\cdot\cdot\cdot$ follows when $h\cdot x$

$$| F2 |\cdot C\cdot\cdot \quad 1h0 \cancel{(tg;t)}\, dt \cdot hbh\cdot\cdot \cancel{2(g\cdot\cdot;t)}\, dt\cdot\cdot\cdot .$$
$$\cdot\cdot h\cdot\cdot\cdot\cdot \cdot \cdot \qquad t \qquad \cdot$$

Thus, when $0\cdot\cdot<2\cdot\cdot$

$$\cdot \qquad h$$
$$| F2 |\cdot C\cdot\cdot\cdot h\cdot\cdot\cdot\cdot 10\cdot\cdot \cancel{(tg;t)}\, dt \cdot hbh\cdot\cdot \cancel{t2(g\cdot\cdot;t)}\, dt\cdot\cdot\cdot\cdot, \cdot = \max(1,\cdot),$$

which completes the proof.

3. Zygmund type estimates for the mixed fractional integral

Theorem 3.1. Let $\cdot\cdot \mathbf{C}(Q)$ and $\cdot(x, y) =|_{x=0,\, y=0} = 0$. Then for (1.1), we have estimates of the Zygmund type

$$\cdot^{1,1}(f; h,\cdot) \cdot C1h^{\cdot}\cdot \cdot \frac{b\, d\, \cdot^{1,1}(\cdot;t,\cdot)}{\quad}2\cdot\cdot\, 2\cdot\cdot\, dtd\cdot. \qquad (3.1)\; h\cdot\, t \cdot$$

$$\cdot^{1,0}(f; h,0) \cdot C_2 h\cdot \frac{b\, \cdot^{1,1}(\cdot; t,d)}{\quad}2\cdot\cdot\, dt,$$
$$h \; t$$

(3.2)

$$\cdot^{0,1}(f; 0,\cdot) \cdot C_3\cdot \cdot \frac{d\, \cdot^{1,1}(\cdot; b,\cdot)}{\quad}2\cdot\cdot\, d\cdot.$$
$$\cdot\;\cdot$$

Proof. Using the identity (2.1), we represent the integral (1.1) in the form

$$\cdot\cdot I\cdot 0\cdot,\cdot,0\cdot\cdot\cdot\cdot\cdot x, y\cdot\cdot \cdot\cdot\cdot \overline{1\cdot\cdot 0,\cdot 0\cdot\cdot x\cdot\cdot 1y\cdot\cdot\cdot\cdot} \quad \overline{x\cdot\cdot\cdot 1\cdot 2\cdot\cdot y\cdot\cdot\cdot y\cdot\cdot 1\cdot\cdot 1} \cdot\cdot x\cdot\cdot\cdot\cdot x, y\cdot,$$
$$\cdot$$

where

$$\cdot 1\cdot x\cdot\cdot \cdot \overset{1\; x}{\overline{\cdot\cdot\cdot\cdot t,0\cdot\cdot 1\cdot\cdot 0,0\cdot dt,\ \cdot 2\cdot y\cdot\cdot \cdot\cdot 1\cdot\cdot\ 0y\cdot\cdot\cdot\cdot 0y,\cdot\cdot\cdot\cdot\cdot 1\cdot\cdot 0,0\cdot d\cdot}},$$
$$\cdot\cdot\ 0 \qquad \cdot x\cdot t\cdot$$

$$\frac{\quad}{\cdot\cdot\cdot 1\cdot\cdot\cdot 0x\cdot\ 0\cdot\ \cdot\cdot} \; y \; \frac{\cdot\cdot \quad \cdot t,\cdot\cdot\cdot\cdot\cdot 0,0\cdot^{\cdot 1,1} \quad \cdot\cdot x, \quad y\cdot}{1\cdot\cdot y\cdot\cdot\cdot\cdot 1\cdot\cdot\, dtd\cdot.\ \cdot x\; t\cdot}$$

Let $h > 0,\, x,\, x \cdot h\cdot[0,b]$. Consider the difference

$$\cdot\cdot\cdot\cdot 1\cdot,0h f\cdot\cdot\cdot\cdot\cdot x, y\cdot\cdot\ \overline{\cdot x\cdot\cdot\cdot h\cdot\cdot\cdot\cdot\cdot\cdot x\cdot}\cdot\ 0y\cdot\ g\cdot\cdot x1,\cdot y.\cdot\cdot\cdot d\cdot\cdot$$

$$\cdot\ \cdot\ \cdot 1\cdot\ \cdot\ \overline{\cdot 0}\cdot^{y}\cdot\ \overline{g\cdot x\ \cdot t,1\cdot y\cdot\cdot\cdot t\ \cdot\cdot hg\cdot 1\cdot\cdot x\cdot,\ y\ \cdot\cdot}\ dtd\cdot\cdot$$
$$\cdot\cdot\cdot\ \cdot\ \cdot h\ 0 \qquad \cdot$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\cdot\ \cdot \frac{1 \quad t\cdot\cdot 1dtd\cdot.}{\quad} \quad \frac{x\, y\ g\cdot x\ \cdot t,\ y\ \cdot\cdot\cdot\ g\cdot x,\ y\ \cdot\cdot\cdot}{1\cdot\cdot} \Bigg|\, \cdot t \cdot h\cdot \quad \cdot\cdot 1 \Bigg| \quad \cdot$$
$$\cdot\cdot\cdot\ \cdot\ 0\ 0 \qquad \cdot$$

The following inequality is valid

$$\Bigg|\ \cdot\cdot\cdot 1\cdot,0h f\cdot\cdot\cdot\cdot \Bigg| x, y\cdot\ \cdot C\cdot\cdot\cdot| (x\cdot h)^{\cdot}\cdot x^{\cdot}\ |\ y^{\cdot}\ 1\frac{\quad}{\quad}\cdot,1(\cdot;1x\cdot,\cdot y\ \cdot\cdot)\ d\cdot\cdot$$
$$\cdot\cdot\cdot \qquad 0\ \cdot$$
$$\cdot$$

$$\cdot\ \cdot\ \cdot \frac{0\, y\ \cdot^{1,1}(\cdot;\ t,\ y\ \cdot\cdot)}{dtd\cdot\cdot\ \cdot h\ 0}(h\cdot t)1\cdot\cdot\cdot 1\cdot\cdot$$

$$\cdot\ \cdot\ \cdot\ \cdot\int_{0\,0}^{x\, y}\cdot^{1,1}(\cdot;\ t,\ y\ \cdot\cdot)|\ (h\ \cdot t)^{\cdot\cdot}1\ \cdot t\cdot\cdot 1\ |\cdot\cdot\cdot 1dtd\cdot\cdot\cdot. \qquad \cdot$$

We make use of (2.2) and obtain

$$\Bigg|\ \cdot\cdot\cdot\cdot 1\cdot,0h\ f\ \cdot \Bigg|\cdot\cdot\cdot x,\ y\cdot\ \cdot\ C^{1}\cdot\cdot\cdot|\ (x\ \cdot\ h)\cdot\cdot\ x\cdot|1\cdot,1(\cdot;\ x,\ d)\ \cdot\ 0\cdot\ \frac{\quad}{\quad}\cdot(h(\cdot\cdot t;)t1,\cdot d\cdot)\ dt\ \cdot$$

$$\cdot\ \cdot\int_{0}^{\cdot h\ x}\cdot\cdot^{1,1}(\cdot;\ t,\ d)\ |\ (h\cdot t)^{\cdot\cdot}1\ \cdot t\cdot^{\cdot}1\ |\ dt\cdot\cdot. \qquad \cdot$$

Using the estimates $\cdot 1,\ \cdot 2,\ \cdot 3$ in the proof of Theorem 2.1, it is easy to obtain

$$|\, f(x, y\ \cdot\cdot) \cdot f(x, y)\, |\cdot\ C2\cdot^{\cdot} \quad \frac{d.c\ \cdot^{1,1}(\cdot;\ b\cdot a,\cdot)}{\quad}2\cdot\cdot\, d\cdot. \qquad (3.3)$$
$$\cdot\ \cdot$$

Similarly, we can obtain the estimate

$$|\, f(x, y\ \cdot\cdot) \cdot f(x, y)\, |\cdot\ C2\cdot^{\cdot} \quad \frac{d\ \cdot^{1,1}(\cdot;\ b,\cdot)}{\quad}2\cdot\cdot\, d\cdot. \qquad (3.4)$$
$$\cdot\ \cdot$$

From (3.3) and (3.4) follows the inequalities (3.2).
Let $h,\cdot>0$ and $x, x\cdot h\cdot[a,b]$, $y, y\cdot\cdot\cdot[c,d]$. Consider the difference

$$\cdot\cdot\cdot\ h,\cdot f\ \cdot\cdot(x, y)^{\cdot 1,1\ \cdot} = \cdot\cdot\cdot\ h,\cdot\cdot\cdot(x, y)^{\cdot 1,1\ \cdot} = \cdot{}^{T}{}_{k} := \cdot\ \cdot\ \cdot\ \cdot \sum_{k=1}^{9}$$

$$:= \frac{\quad}{\cdot(1\cdot\cdot)\cdot(1\cdot\cdot)} g(x, y)^{\bullet}(x\cdot h)^{\cdot}\cdot x^{\cdot\bullet}[(y\ \cdot\cdot)^{\cdot}\cdot y^{\cdot}]\ \cdot$$

$$\frac{(y\ \cdot\cdot)^{\cdot}\cdot y^{\cdot}}{\cdot\ \cdot(\cdot)\cdot(1\cdot\cdot)}\int_{\cdot h}^{0}\frac{g(x\ \cdot t, y) \cdot g(x, y)}{1\cdot\cdot}\ dt\ \cdot$$

(t · h)

(x · h)˙· x˙ 0 g(x, y ··) · g(x, y)

· (1) ( ) · 1·· d··

· ··· · ·· (···)

· (y ··)˙ · y˙x··g(x ·t, y) · g(x, y)••(t · h)··1 ·t··1•dt ·

·(·)·(1··) 0

· (x · h)·· x· y.•g(x, y ··) · g(x, y)••(···)··1 ····1•d··

·(1··)·(·) 0

·1,1 ·

···t,··g·(x, y)

· ( )1 ( ) 0· 0· ·· 1··(·····)1·· dtd··

··· · ·h··(h·t)

·1,1 ·

· ( )1 ( ) 0˙ y· ˙· ·t,··g1··˙·(x, y) •(···)··1

··˙·1•dtd·· ··

··· · ·h0 (h·t)

·1,1 ·

· · ·t,··g·(x, y)

1 x 0 · ·

· ·(·)·(·) 0·..· · (···)1··· •(t · h)··1 ·t··1•dtd··

1 x y·1,1 ·

· ( ) ( ) · ·····t,·· g··(x, y)•(t · h)··1 ·t··1•·

··· · 0 0· ·

•(···)··1 ····1•dtd·.

The inequality is valid

·1,1 · ·1,1

··h,· f ·(x, y) · C··(·; x, y) | (x · h)˙· x˙ || (y ··)˙· y)˙ | ·

·· · ··

·

1,1 1,1

· | (y ··)˙ · y˙ | h˙ ·(·; t1,··y) dt· | (x · h)· · x· |·· ·(·; x1,···) d··

0 (h ·t) 0 (···) x1,1

· | (y ··)˙ · y˙ | ··(·; t, y) | (h · t)··1 ·t··1 | dt ˙

0 y·c1,1

· | (x · h · a)˙· (x · a)˙ | · ·(·; x · a,·) | (···)··1 ··˙·1 | d··

0

1,1

h· ·(·; t,·)

· 0· 0· (h ·t)1··(···)1·· dtd··

h y1,1

˙ · · ·(·; t,·)(h ·t)··1 | (···)··1 ··˙·1 | dtd··

0 0

x ·1,1

˙ · · ·(·; t,·) | (h · t)··1 ·t··1 | (···)··1dtd··

0 0

x y1,1 ··1 t··1 || (···)··1 ··˙·1 | dtd· ··.Each ˙ · ··(·; t,·) | (h · t) ·

0 0 ·

term of this inequality is estimated in the standard way, and one can obtain

1,1

| ···1,·1h,˙ f | ···(x, y) · C3h·b· d· ·2·(··;

t2,··.) dtd·,

· ·h ·t · from which inequality (3.1)

follows.

Theorem 3.2. Let ·(x,y) =··x···y··x˙y˙,0··< 2··, 0 ··< 2 ··. If the function ·(x, y)·Q satisfies the following conditions:

1) ·0(x, y) = ·(x, y)·(x, y)·**C**(Q) and ·0(x, y)|x=0, y=0= 0;

1,1

2) b· ad· c·(· 0; t,·) dtd· the integral converges for

0 0 t ·

· = max{1,·},·= max{1,·} . Then the following estimates of Zygmund type are valid

1,0 ˙· ····1h1·,1(··;t,d) dt·

·(f ;h,0) ·C1·h ·t·

0

·

·

1,1 ·

· hb· ·(t··2··;t,d) dt··· , (3.5) h

·

0,1 ˙ ·1·,1(··;b,·)

·(f ;0,·) ·C2·····1˙ · d··

· 0 ·

·

1,1

··d· ·(··2·;·b,·) d···· , (3.4)

· · ·

·

1,1 ·· ····1 ····1h·1·,1(··;t,·) dtd··

·(f ;h,·) · C3·h · · · ··

· 0 0 t ·

·

1,1 1,1

· h·····1^b h· ˙0· __________·(2····;t·,·) dtd·· h.···1·^h. d. __________

·t·(···2·;t·,·^) dtd·· t · 0 ·

1,1 ·

· h·b· d· __________·t2·(····;2t·,··) dtd····. (3.5)

h·

·

Proof. By Remark 2.1, it suffices to deal with the case ·,··1.

Let ··H͂˙0 ···, so that ·0·x, y····x, y···x, y·, where

~˙··· and ·0·x, y· x·0, y.0 · 0 . For

·0·x, y··H0

|

G·x, y·:· 0^x· 0^y˙··t,·····xx,·y··t·10··t,··y·dtd..··1·· .

We represent G·x, y· in the form

G(x, y) = ___1___ ··x y ·0(t,·)dtd· ·

·(·)·(·) ··0· 0· (x ·t)1··(y ··)1··

x y ·

· · .B(x, y;t,·)·0(t,·)dtd·˙. = G1(x, y) ·G2(x, y).

0 0 ·

Here the question of the estimation of the modulus of continuity for the first term is solved by us in Theorem 3.1. Therefore, inequalities

·(^G 1;h,0) ·^C 1·h˙·(·0;h,d) ·^h. ··2^0.. dt·, __________ (3.6)

· h t ·

· ·

0,1 ·· ·1,1(·0;b,·) ··d· 1·,1(·0;b,·) ~~d····,~~ ·(^G 1,0,·) ·^C 2·· · (3.7)

1,0 ·· 1,1 b1,1(;t,d) ··

2··

· .· ·

· ·

1·,1 ·· ··1,1(·0;h,·) · h··b· 1·,1(t·20··;t,·) dt· (G1~~;h,·) ·C3·h·~~ ·

· h

·

1,1 1,1 ·

d

· h···· ··(~~·20·;·h,·) d~~·· h·bh· ·d· t·2·(~~···0;2t·,··) dtd~~····. (3.8)

·

To estimate the term G2·x, y·, we note that the weight being degenerate, we have

··x, y····t,·· [··x····t·][··y·····]·····[··x····t·]·

,

···t·[··y·····] which leads to the following representation

x y

G2·x, y·· · ·B1·x,t·B2·y,··0·t,··dtd··

0 0

x y

· · ·B1·x,t··0·t,···y ···^··1 dtd··

0 0 x

y

· · ·B2·y,···0·t,···x ·t·^··1 dtd·.

0 0

where the notation (2.3) has been used. For the difference

·1,0 ·

···hG2 ··x, y· with h · 0 and x, x · h··0,b·, we have

· ·

·1,0 · x·h y

·· ·hG2 ··x, y·· · ·B1(x · h,t)B2(y,·)·0(t,·)dtd··

· · x 0

x y

· . .D1(x,h,t)B2(y,·)·0(t,·)dtd··

·x0 ··0h y.B (x · ~~h,t) ·0(t,·) dtd~~··

··,··0·t,··d·dt ·

1

x 0 (y ··)1··

· 0x· 0y.D1(x,h,t) (y··0·(t),·) dtd·· x·x·h 0y· ·0(t,·)B21(·y·,·) dtd··

1·· (x · h·t)

x y

· 0· 0··0(t,·)•(x · h ·t)··1 · (x ·t)··1•B2(y,·)dtd·.

Since ·0·x,0·· 0 then the inequality

·1,0 ·x·hy

··,·)·0(t,·)dtd··

····h G2 ····x,xyy·· x· 0. | B1(x · h,t) ||1B,12(y,·) | ·(·0;t,·)dtd··

· · . | D1(x,h,t) || B2(y,·) | ·(·0;t,·)dtd··

0 0

x·x·h 0y· | B1(x · h,t)1| ,1(·1,y1(···0);1t·,··) dtd··

0 0

x··x·hx· 0y·y· |(xD1·,1·1(~~(·hx,0·h;,ttt),~~)1··|)··(y|(· B 02;)(1ty,··,··))dtd| dtd····

xy1,1

· 0· 0··(·0;t,·) | (x·h·t)··1 ·(x·t)··1 || B2(y,·) | dtd·.

We make use of (2.2) and obtain

x·h 1,1

| G2(x · h, y) ·G2(x, | y)· . | B1(x · h,t) | ·(·0;t,d)dt · · ·

x 1,1 x·h 1,1

· . | D1(x,h,t) | ·(·0;t,d)dt · . | B1(x · h,t) | ·(·0;t,d)dt ·

0 x

·1,1 ·

For the mixed difference ··h,.G2 ··x, y· with h,·· 0 ε

· ·

x,x · h·[0,b], y, y ···[0,d] the appropriate representat

leading to the separate evaluation in each variable with

losses in another··1·,1h,·G2 ····x, yvariable is as follows:·· x··hy···B1·x · h, t·B2 ·y

·· · x y

·x· y·D1(x,h,t)D2(y,·,·)·0(t,·)dtd··

0 0

· x··h y·B1(x·h,t)D2(y,·,·)·0(t,·)dtd··

x 0
x y··

1,1 · · . D1(x,h,t)B2(y

·0 x··yh y··· B1(x · h,t) ·0(t,·)dtd··

(y ····)1··

x y

x·h y

· x· 0··B1x·(xy···.·hD,t1)(•(xy,h·,·t)(·y··)····1··)·(y·1···0)(·t,··1)•·dtd0(·t,··) dtd··

D1(x,h,t) (y ····) .

· 0x·· 0y·x··0h yy···(x• · h·t)·· 1B 21(y(·y··,··))· 0 1(t•·,·0)(dtdt,··)dtd···

x y

·x· y···•(x · h·t)··1 · (x ·t)··1•B2(y ··,·)·0(t,·)dtd··

0 y

x·h y

·(x · h ·t)··1D2(y,·,·)·0(t,·)dtd·· x x 0

x y

· · · •(x · h ·t)··1 · (x ·t)··1•D2(y,·,·)·0(t,·)dtd·.

0 0

· · | D1(x,h,t) | ·(·0;t,d)dt · . (x · (·h 0· ·t ;t, ·t) 1 d) · . dt ·

0 x x1,1

· ··(·0;t, d) | (x · h ·t)··1 · (x ·t)··1 | dt .

0

From the estimates ·1, ·2, ·3 of Theorem 2.1 and from the estimates F1, F2 in Theorem 2.2, one can easily verify the validity of inequality

1,1

...1,·0hG2 ···x, y· · C1·· h····1h· ·(··t·;t,d) dt

· · · 0

1,1

b.a ·(··;t,d · c) ·

· h · t2.. dt ··, (3.9) h

где ·= max(1,·) . The estimate

····0·,1·G2···· ····1·1·,1(··· ;b,·) d· ···d· ·,1(···2·;·b,·) d·····,

·x, y··C2·· · ·

·· 0

· ·

is symmetrically obtained, where ·= max(1,·) .

···h,.G2 ··(x, y) ·1,C · ·x·hy·· · B1(x · h,t)B2(y ··,·)·

··x y

1,1 x y 1,1

··(·0;t,·)dtd·· · · D1(x,h,t)D2(y,·,·)·(·0;t,·)dtd··

0 0

x·h y 1,1

· · · B1(x·h,t)D2(y,·,·)·(·0;t,·)dtd·· x 0

x y·· 1,1

· · · D1(x,h,t)B2(y ··,·)·(·0;t,·)dtd··

0 y

·x··h y··· B1(~~x · h,1t)·1·,1(·~~0;t,·)dtd·· x y (y ····)

x·h y 1,1

· · ·B1(x · h,t)•(y ····)··1 · (y ··)··1•·(·0;t,·)dtd·· x 0 (3.10) x y··

1,1

· · · D1(x,h,t)(y ····)··1·(·0;t,·)dtd ··

0 y

x y 1,1

· · ·D1(x,h,t)•(y ····)··1 · (y ··)··1•·(·0;t,·)dtd··

0 0

x·h y·· 1,1

· · · (x · h·t)··1B2(y ··,·)·(·0;t,·)dtd·· x y

x y·· 1,1

· · · •(x · h·t)··1 · (x ·t)··1•B2(y ··,·)·(·0;t,·)dtd··

0 y

x·h y 1,1

· · ·(x · h ·t)··1D2(y,·,·)·(·0;t,·)dtd··

x 0

x y 1,1

· · ·•(x · h·t)··1 · (x ·t)··1•D2(y,·,·)·(·0;t,·)dtd·.

0 0

We omit the details of evaluation of each term in the above representation; it is standard via Lemma 2.1 and yields

·1,1 · ·· ····1 ····1h·1·,1(··;t,·) dtd··

····h,·G2···(x, y) ·C3··h · 0· 0· t···

·

1,1 1,1

·h·····1bh· ·0· ·t2(·····;t·,·) dtd··h····1·h0··d· ·t·(···2;·t·,·) dtd··

1,1 ·

· h·b· d· ·t2·(····;2t·,··) dtd····, (3.11)

h ·

·

where·= max(1,·) и ·= max(1,·) .

From the inequalities (3.11), (3.10), (3.9) and (3.6), (3.7), (3.8), we obtain the corresponding estimates (3.3), (3.4) and (3.5).

4. Mapping properties of the mixed fractional integration operators in the space $\tilde{H}^{\cdot}_{0}$ ·,·

In this section, we give a generalization of the theorem to the weighted.

Theorem 4.1. Let 0 <·,·<1, ·(x,y) = (x·a)·(y·c)·, 0 ··< 2 ··, 0··<2··. If ·(x, y)··(Q) and assume that

1,1

1) x· y·· x ·····y ····t,··dtd·· C1·,1·x, y·,

0 0· t · ··· t·

2) b· d··· x ··1·····y ··1··1·1·t,··dtd·· C1·,1·x, y·,

x y· t · ··· t·

where ·= max(··1,0), ·= max(··1,0) . Then the mixed fractional integral operator $I^{\cdot,\cdot}_{0\cdot,0\cdot}$ is bounded from the weight space $\tilde{H}^{\cdot}_{0}(\cdot)$ to the space $\tilde{H}^{\cdot}_{0}$··(·) with the same weight and with the characteristic ·,,,(t,·) = t·,·(t,·) .

∵ , where $\cdot\cdot H\tilde{}_0\,(\cdot)$. We

Proof. Let $f = I_{0,0}\cdot$

$\sim\cdot\therefore(\cdot)$. For this, it suffices to show that will show that $f \cdot H_0$

1,0 0,1

$h\sup>0$ ——— $\cdot(\cdot\cdot f;h,0) = C_1 < \cdot$, sup ———

$\cdot(\cdot\cdot f;0,(\cdot\cdot)) = C_2 < \cdot$, $h \cdot 1(h) \cdot>0 \cdot\cdot 2$

1,1
$\cdot(\cdot f;h,\cdot)$
———
$h>\sup 0,\cdot>0 \cdot\cdot (h,\cdot) = C_3 < \cdot$.
$h \cdot\cdot 1,1$

From membership $\cdot(t,\cdot)$ in the class $\cdot(Q)$ and satisfaction of inequalities (4.1), (4.2) the convergence of the integrals follows

1,1 1,1 1,1
b ~~$\cdot(\cdot\cdot;t,d)$~~ d $\cdot(\cdot\cdot;b,\cdot)$ $b\ d$ ~~$\cdot(\cdot\cdot;t,\cdot)\ 0\cdot\ t$~~ dt, $0\cdot \cdot\cdot d\cdot$, $0\cdot\ 0\cdot$
——— $t\cdots dtd\cdot$.

Therefore, there are estimates of the Zygmund type from Theorem 3.2. Whence follows

$1\cdot,0(\cdot f;h,0)$ $\cdot\ \cdot\cdot 1\ h1\cdot,1(\cdot\cdot;t,d)$
$\cdot h$
———
$h\dot{}\cdot 1,1(h,d) \cdot C\ 1\cdot\cdot\dot{}1,1(h,d)\ 0\cdot$ $t\cdot$ $dt\cdot$
.
1,1 $\cdot h1\cdot\cdot b\cdot(\cdot\cdot;t,d)$
.
———
$\dot{}1,1(h,d)\ h\cdot$ $t2\cdot\cdot dt\cdots$
.

$h\ dt\cdot$ $\cdot\ 2\cdot\cdot dt\cdot$.
$\cdot C1 \cdot\cdot H\!\!\|\!\sim\cdot 0\|\cdots 1,1$ ~~$\cdot(h\cdot 1,d)\ h0\cdot$~~ ~~$1\cdot,1(t\cdot t,d)$~~ $\cdot 1$~~$h,11(\cdot h\cdot,d)$~~ ~~$bh1\cdot,t1(t$~~$,d)\cdots$

. .

$0\cdot,1(\cdot f;0,\cdot)$ $\cdot\cdot\ \cdot\dot{}1\ \cdot 1\cdot,1(\cdot\cdot;b,\cdot)$
———
$\therefore 1,1(b,\cdot) \cdot C2\cdots 1,1(b,\cdot)\ 0\cdot\ \cdot\cdot d\cdot\cdot$
.

$1\cdot\cdot\ d\ 1\cdot,1(\cdot\cdot;b,\cdot)$ $\cdot\cdot$
$\cdot\ \cdot\cdot 1,1\ \cdot$ ——— $2\cdot\cdot d\cdots$
$(b,\cdot)\cdot\ \cdot$
.
$\cdot\ \cdot$
$\cdot C2 \cdot\cdot H\!\!\|\!\cdot 0 \cdot\|\cdot\cdot 1,1\cdot$~~$(b\cdot,1\cdot)\cdot 0\cdot\ 1\cdot,1\cdot(b\cdot,\cdot)\ d$~~$\cdot\cdot\ \cdot 1\cdot,$~~$11(\cdot b\cdot,\cdot)\cdot d\cdot$~~ 1~~$\cdot\cdot,12(b\cdot,\cdot$~~$\cdot)\ d\cdots\cdots$.
. .

$1\cdot,1(\cdot f;h,\cdot)$ $\cdot\cdot\ h\cdot\cdot 1\cdots 1\ h\cdot 1\cdot,1(\cdot\cdot;t,\cdot)$
——— $\cdot C$ ——— ———
$h\dot{}\cdots 1,1(h,\cdot)$ $3\cdots 1,1(h,\cdot)\ 0\cdot\ 0\cdot$ $t\cdots dtd\cdot\cdot$
.
1,1 1,1
$h\cdot\cdot 1\cdot 1\cdot\cdot\ h\ d\ \cdot(\cdot\cdot;t,\cdot)$ $h1\cdots\cdots 1\ b\cdot\cdot(\cdot\cdot;t,\cdot)$
$\cdot\ \cdot 1,1(h,\cdot)\ 0\cdot.\cdot$ $t\cdot\cdot 2\cdot\cdot\ dtd\cdot\cdot\ \cdot 1,1(h,\cdot)\ h\cdot\ 0\cdot$ $t2\cdots\ dtd\cdot\cdot$
1,1 $\cdot\ h1\cdots 1\cdot\cdot\ b\ d\ \cdot(\cdot\cdot;t,\cdot)$ $\cdot$
$\cdot\ \cdot 1,1(h,\cdot)\ h\cdot\ \cdot\cdot\ t2\cdots 2\cdot\cdot\ dtd\cdot$(4.1)$\cdots$

$\cdot\ \cdot\cdot 1\ \cdot\cdot 1\ h\cdot 1\cdot,1(t,\cdot)$
$\sim\cdots h\ \cdot\ \cdot\cdot\ \cdot\ \cdot$ (4.2)$dtd\cdot\ \cdot$
$\cdot\ C3\ \cdot\cdot\|H0\|\cdot\cdot 1,1(h,\cdot)\ 00\ t\ \cdot$
.
1,1 1,1
$h\cdot\cdot 1\cdot 1\cdot\cdot\ h\ d\ \cdot(t,\cdot)$ $h1\cdots\cdots 1\ b\cdot\cdot(t,\cdot)$
$\cdot\ \cdot\ (h,\cdot)\ 0\cdot\ \cdot\ t\cdot\ 2\cdot\cdot\ dtd\cdot\cdot$ 1,1 $\cdot\ \cdot\ t2\cdots dtd\cdot\cdot$
1,1 $\cdot\ \cdot$ $\cdot(h,\cdot)\ h\ 0$
1,1 $\cdot\ h1\cdots 1\cdot\cdot\ b\ d\ \cdot(t,\cdot)$ $\cdot$
$\cdot\ 1,1(h,\cdot)$ $h\ \cdot\ \cdot\ t2\cdots 2\cdot\cdot\ dtd\cdots$.
. .

It's obvious that

1,0 0,1
$\cdot(\cdot f;h,0)$ $\cdot(\cdot f;0,\cdot)$
~~$h\cdot\cdot 1,1(h,d)\ H$~~$\cdot 0\ \cdot\cdot\cdot\|\cdot\cdot\|1,1(b,\cdot)\cdot C$~~$2\cdot H\sim\cdot 0\cdot\cdot\cdot,\cdot$~~ $C1\cdot\|\ \|$
1,1
$\cdot(\cdot f;h,\cdot)$
——— $\cdot C$ $\|\ \|$
$h\cdots 1,1(h,\cdot)$ $3\cdot H\sim\cdot 0\cdot\cdot\cdot$. (4.3)

We estimate $\cdot\|f\|_{C(Q)}$. We have

$\cdot(x,y)\,f(x,y) = x\cdot y\cdot 0x\cdot\ 0y\cdot\ t1$~~$\cdots 01(\cdot x\cdot\cdot(tx,y\cdot\cdot t)\cdot\cdot)(dtdy\cdots$~~$)\cdot =$ [10] weight and operators of B.S. Armyan. SSR Ser. Mat. 9, No 4. (1974), pp. 308Rubyn. Fractional integralspotential in Hölder spaces with type. Izv. Akad. Nauk -324.

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Since $\cdot_0(x,y)\,|_{x=0,y=0}= 0$, then Nauk Armyan. SSR Ser. Mat. 21, No 5 (1986), pp. 488503
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0,1 Gordon and Breach. Sci. Publ., N. York - London, 1993, ·0•x · x·, y · ys•··0•x · x·, 0• ·
C2 ·(·0;1· s), 1012 pp. (Russian Ed. - Fractional Interals and

·1,1 ·1,1 Derivatives and Some of Their Applications, Nauka i
·x(1··),y(1·s)·0·(x, y) · C3·(·0;1··,1· s). Tekhnika, Minsk, 1987).

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It follows that estimates for fractional differentiation and integration and
1 1 ·1,1(t,·)dtd· their applications. Trudy. Matem. Inst. Steklov 180

| ·(x, y) f (x, y) |· C H~·0 (·)0· 0· t···(1·t)1··(1··)1·· . (198Math. (AMS) 1989, Issue 3 (1989),7), 197-198; English transl. in: Proc. Steklov Inst. pp. 233-235.

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From the inequalities (4.3) and (4.4) follows the assertion of the theorem.

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