Computer Engineering **Volume 14 | Issue 5 | Year 2025**
ISSN: 10003428 https://journalofcomputerengineering.com/

Maximal Subgroup Structure in Countably Categorized Groups

Anastasios Kontogiannis, Sofia Papadopoulou, and Ethan Fraser

Anastasios Kontogiannis and Sofia Papadopoulou, Department of Mathematics, University of Thessaly, Volos, Greece; Ethan Fraser, Department of Pure Mathematics, University of Aberdeen, Aberdeen, Scotland, UK

Abstract. The object of this work is to find classes of groups which possess only countably many maximal subgroups. Modules with countably many maximal submodules and group rings having countably many maximal right ideals are also investigated. Examples of soluble groups with uncountably many maximal subgroups are described.

1 Introduction

This article is a study of groups that have only countably many maximal subgroups, a property that is denoted here by CG. There are many groups with CG; indeed, a group with no maximal subgroups at all has the property and little can be said about such groups in general. On the other hand, non-trivial finitely generated groups always have maximal subgroups and clearly the cardinality of the set of maximal subgroups is at most $2@^{0}$. In fact, finitely generated groups with uncountably many maximal subgroups exist even in the soluble case, as has been shown by de Cornulier [3]: thus for a finitely generated group to be a CG-group is a real restriction on the group structure.

As examples of CG-groups we cite virtually soluble groups with finite abelian ranks and finitely generated nilpotent-by-polycyclic-by-finite groups; in each case the reason is that in these groups maximal subgroups have finite index and there are finitely many subgroups of each finite index. (For the background here see [8, Chapters 5 and 7].) We mention as well the celebrated theorem of Margulis and Soifer [9] that a finitely generated linear group over a field which has CG is virtually soluble. The converse statement is also true since virtually soluble linear groups are nilpotent-by-abelian-by-finite by a well-known theorem of Mal’cev.

In a CG-group the maximal subgroups always have countable index (Lemma 2). In the present work we will find many diverse types of CG-groups with maximal subgroups of infinite index. In the final section we describe methods for constructing finitely generated soluble groups with uncountably many maximal subgroups.

Results. We begin with a discussion of extension properties of the class CG. The first significant result states that a finite extension of a CG-group is a CG-group (Theorem 1). A further extension theorem which allows us to expand our list of groups with CG is:

Theorem 2. *Let* G *be a group with a normal subgroup* N *which has countably many subgroups and let* G=N *be finitely generated. Then all but a countable number of maximal subgroups of* G *contain* N*.*

Thus, if in addition G=N has CG, then G has CG. For example, a finitely generated extension of a group with max by a soluble group of finite rank has CG. In general, a direct product of CG-groups need not be a CG-group; however, we are able to establish:

Theorem 3. *Let* H *and* K *be groups with* CG *and assume that simple quotients of* K *are finitely generated. Then* H K *has* CG*.*

We are particularly interested in classes of locally nilpotent groups and soluble groups which have CG. In the case of periodic hypercentral groups there is a precise characterization.

Theorem 5. *Let* G *be a periodic hypercentral group. Then* G *has* CG *if and only if it is an extension of a central divisible subgroup by a direct product of finite* p-*groups for distinct primes* p*.*

The relation between soluble groups satisfying chain conditions on normal subgroups and the property CG is especially interesting. It is shown in Theorem 6 that *a soluble group satisfying* max-n *has* CG*.* On the other hand, the status of soluble groups with min-*n* is unknown, mainly due to our lack of knowledge of such groups. Our strongest result in this direction is the technical Theorem 7 below. A special case of interest is: *metanilpotent groups satisfying* min-n *have* CG. This seems to be the only structural property of metanilpotent groups with min-n that has been found since the work of Hartley and McDougall over forty years ago; see [6,7,10].

Further investigations of CG-groups are aided by introducing the corresponding module property: a module has the property CM if it has countably many maximal submodules. In the same spirit we will say that a ring has the property CI if it has countably many maximal right ideals. It is an easy exercise to show that a countable ring R with identity has CI if and only if there are only countably many R-isomorphism types of simple modules. The case that concerns us here is that of group rings; of course a group ring ZG has the property CI precisely when it has CM as a G-module.

While the structure of soluble groups with CG seems beyond reach in the general case, there is a characterization for metanilpotent groups.

Theorem 8. *Let* G *be a metanilpotent group and let* A G G*, where* A *and* G=A *are nilpotent. Then* G *has* CG *if and only if the following conditions hold:*

(i) $G=G^pG_0$ *is finite for each prime* p*.*

(ii) *Each maximal* G*-submodule of* A^{ab} *has countable index in* A^{ab}*.*

(iii) *There are countably many maximal* G*-submodules of* A^{ab} *whose quotients in* A^{ab} *are not* G*-trivial.*

A large class of finitely generated CG-groups is given by:

Theorem 9. *Let* G *be a finitely generated group with a normal subgroup* N*. Assume that* N *has an ascending* G*-invariant series in which all the infinite factors are abelian and have the property that their finitely generated* G*-submodules satisfy* CM*. Then all but a countable number of maximal subgroups of* G *contain* N*.*

Of course, if in addition G=N has CG, then so does G. The property CI is clearly of interest in its own right. It has drastic structural consequences for soluble groups, as the next result shows.

Theorem 10. *If* G *is a soluble group whose group ring* ZG *satisfies* CI*, then* G *is a minimax group with no sections of type* $p^1\ p_1$*.*

It is interesting that the groups described in Theorem 10 are exactly the soluble groups with countably many subgroups, as was shown by Cutolo and Smith [4]. The converse of Theorem 10 is established for abelian-by-finite groups (Theorem 11). Theorems 10 and 11 have a noteworthy consequence: *if* G *is a locally finite group, then* ZG *satisfies* CI *if and only if* G *is a Cernikov group whoseˇ maximum divisible subgroup is locally cyclic*. This is Corollary 4.

The methods of proof used in the article are mostly standard, a common tool being the uncountable pigeonhole principle. However, derivations play an important part, while the proofs of Theorems 7 and 8 call for the use of cohomological vanishing theorems of the third author.

Notation. In what follows, we will use the following notation.

(i) CG: a group has countably many maximal subgroups.

(ii) CM: a module has countably many maximal submodules.

(iii) CI: a ring has countably many maximal right ideals.

(iv) max-G, min-G: the maximal and minimal condition for G-invariant subgroups.

(v) G^{ab}: the abelianization of G.

(vi) $G^{.i/}$, $_{i}.G/$: terms of the derived series and lower central series of G.

(vii) H_G: the G-core of H.

(viii) All modules are right modules. If G is a group, a G-module is a ZG-module.

(ix) Der.G;A/, Inn.G;A/: sets of derivations and inner derivations.

(x) I_G;IN_G: an augmentation ideal and a relative augmentation ideal.

2 Some groups with countably many maximal subgroups

We begin with two elementary results.

Lemma 1. *Let* G *be a group whose maximal subgroups have finite index and assume that* $G=G^n$ *is finite for all* $n > 0$*. Then* G *is a* CG*-group*

Proof. A maximal subgroup contains G^n for some $n > 0$, so there are only countably many of them.□

Therefore finitely generated nilpotent-by-polycyclic groups and soluble groups with finite abelian ranks are CG-groups. On the other hand, finitely generated soluble CG-groups with maximal subgroups of infinite index exist; for example, the soluble group with max-n constructed by Hall [5] – see also [8, 4.5.1]. The next result is almost the only property shared by all CG-groups.

Lemma 2. *If* G *is a* CG*-group, every maximal subgroup of* G *has countable index.*

Proof. Let M be a maximal subgroup of G. If M G G, then jG W Mj is finite. Otherwise M D N_G.M/, so that M has jG W Mj conjugates in G. Since each conjugate is maximal in G, the result follows. □

In the remainder of this section we investigate extension properties of the class of CG.

Theorem 1. *Let* G *be a group with a normal subgroup* N *of finite index. If* N *has* CG*, then so does* G.

Proof. Assume the result is false; then there are uncountably many maximal subgroups M of G such that N 6 M. Also G D MN, D D M \ N G M and the quotient M=D is finite. In addition, D is a maximal M-invariant subgroup of N.
For suppose that D < L N and L D L^M; then we have L 6 M, so G D ML and N D .ML/ \ N D DL D L.

Next let T be a transversal to D in M; thus M D DT . Let x 2 NnD; then hD;x^t j t 2 T i is M-invariant and contains D properly, so that N D hD;x^t j t 2 T i by maximality of D. Thus N is generated by D and finitely many elements, which implies that D is contained in a maximal subgroup X of N. Moreover, D D X_M D X_T by maximality of D. Therefore D is the intersection of finitely many maximal subgroups of N each of which has countable index in N by Lemma 2. Consequently, jG W Dj is countable. Since N has CG, there are countably many possible subgroups D. Finally, M is a finite union of cosets of D, so there can be only countably many subgroups M, a contradiction. □

On the other hand, it is uncertain whether the property CG is inherited by normal subgroups of finite index.

Theorem 2. *Let* G *be a group with a normal subgroup* N *which has only countably many subgroups and let* G=N *be finitely generated. Then all but a countable number of the maximal subgroups of* G *contain* N.

Proof. Assume the result is false, so there are uncountably many maximal subgroups M of G such that N 6 M. Thus G D MN and M \ N G M. By hypothesis there are countably many possibilities for M \ N. Hence M \ N D D is fixed for uncountably many subgroups M. Notice that D G hM;M_1i D G, where

M_1 is another maximal subgroup with the properties of M. Now we consider G=D D .M_0=D/ Ë .N=D/ with M_0 a fixed maximal subgroup. It is well known that complements of N=D in G=D correspond to elements of Der.M_0=D;N=D/. A derivation is determined by its effect on the generators of M_0=D. Since M_0=D ' G=N is finitely generated and N is clearly countable, there are only countably many quotients M=D and hence subgroups M, a contradiction. □

Corollary 1. *A finitely generated extension of a group with max by a* CG*-group is a* CG*-group.*

Nevertheless, the class CG is not closed under extensions. For example, consider the metabelian group G D H Ë K, where K is the additive group and H the multiplicative group of the field of complex numbers, with the natural action of H on K. Both H and K have CG since they are divisible and abelian. However, H is maximal in G and has uncountable index; thus it G not a CG-group by Lemma 2. We turn next to direct products of CG-groups.

Theorem 3. *Let* H *and* K *be* CG*-groups and assume that all simple quotients of* K *are finitely generated. Then* H K *is a* CG*-group.*

Proof. Write G D H K and assume that G does not have CG. Then there are uncountably many maximal subgroups M that contain neither H nor K; hence

G D MH D MK and M \ H G G;M \ K G G. Also M \ H and M \ K are maximal normal in H and K, respectively. By hypothesis K=M \ K is finitely generated, so M \ K is contained some maximal subgroup K.M/ of K. If MN is another maximal subgroup like M and K.M/ D K.M/N , then M \ K K.M/N and, since K=MN \ K is simple, it follows that M \ K MN \ K and hence we have M \ K D MN \ K by maximality. There are only countably many subgroups

M \ K because K is a CG-group. It follows that there are uncountably many maximal subgroups M with M \ K D L fixed. Factoring out by L, we can assume that M \ K D 1 for uncountably many M. Thus K is finitely generated.

Next G=M \ H D H=.M \ H/ .M \ H/K=.M \ H/, so that

H=M \ H ' G=.M \ H/K D MK=.M \ H/K ' M=M \ H ' K:

As a consequence H=M \ H is finitely generated. Arguing as we did above for K, we may also suppose that M \ H D 1 and H is finitely generated. Now the situation is that G D M Ë H D K H. The complements of H in G correspond to elements of Der.K;H/ D Hom.K;H/, which is countable since H and K are finitely generated. This gives the contradiction that there are countably many subgroups M. □

In particular, the direct product of two CG-groups is a CG-group if at least one of them is soluble. On the other hand, the direct product of two insoluble CG-groups need not be a CG-group. Indeed, suppose that S is a simple group without maximal subgroups: we could take S to be, e.g., Ol'šanskiˇi's uncountable group with all its proper subgroups countable [11, 35.2–35.4]. Set G D S S and observe that the diagonal subgroup is maximal and has uncountable index in G; therefore G cannot be a CG-group by Lemma 2.

3 Locally nilpotent groups and soluble groups with the property CG

In this section we investigate the property CG in the context of locally nilpotent groups and soluble groups. For locally nilpotent groups there is a simple – if not too informative – characterization.

Lemma 3. *Let* G *be a locally nilpotent group. Then* G *has* CG *if and only if* $G^{ab}=.G^{ab}/^{p}$ *is finite for all primes* p.

Proof. Recall that a maximal subgroup of G is normal with prime index and hence contains some G_0G^p. Also an infinite elementary abelian p-group has uncountably many subgroups of index p. The result now follows. □

The structure of even abelian groups with CG is not known in general. Clearly an abelian group with finite ranks and the direct product of a divisible abelian group and finite abelian p-groups for distinct primes p are CG-groups. But there are uncountable, torsion-free reduced abelian groups of infinite rank with CG, for example the cartesian product of the additive groups of rational numbers with denominators indivisible by a prime p for distinct p.

Turning to soluble groups satisfying CG, we will find several uses for the next result.

Theorem 4. *Let* G *be a virtually soluble group of finite exponent. If* G *has* CG*, then it is finite.*

Proof. Let N be a soluble normal subgroup with G=N finite and let d be the derived length of N; we argue by induction on d > 0. Write A D $N^{.d\ 1/}$; then by induction G=A is finite. Assume the result is false; then A is infinite and hence $A=A^p$ is infinite for some prime p, since A is reduced. Thus we may suppose that A is an infinite elementary abelian p-group. Next G D XA, where X is a finite subgroup and X \ A G XA D G, so we can factor out by X \ A and assume that G D X Ë A. If M is a maximal X-submodule of A, then XM is maximal in G. It follows that there are only countably many such subgroups XM and therefore A has countably many maximal submodules, i.e., it has CM as an X-module.

To reach a contradiction, the first step is to show that there are uncountably many X-submodules with finite index in A. Certainly there are uncountably many subgroups B such that jA W Bj D p. Then B_X is a submodule of finite index in A. Suppose that there are only countably many of these B_X. Then B_X D C is fixed for uncountably many B. But A=C is finite and C B < A, which leads to the contradiction that there are finitely many subgroups B.

Let S_n denote the set of all submodules S such that A=S is finite with X-composition length n. By hypothesis there is a least n > 0 such that jS_nj > $@_0$. If S 2 S_n, then A=S has composition length n and hence S is a maximal submodule of some submodule belonging to $S_{n\ 1}$. Since $S_{n\ 1}$ is countable whereas S_n is uncountable, there exists B 2 $S_{n\ 1}$ with uncountably many maximal submodules. Also A=B is finite, so we have A D BD for some finite X-submodule D.

Let I D B \ D and consider A=I D B=I D=I. If S=I is a maximal X-submodule of B=I, the submodule SD is maximal in A, so there are countably many submodules SD. It follows that there are uncountably many maximal X-submodules M of B not containing I. For such an M we have B D MI and, as I is finite, there are uncountably many such submodules M with M \ I D J fixed. Since B=J D M=J I=J, we see that H D Hom$_X$.B=I;I=J/ must be uncountable. Let 0 ¤ 2 H. Now I=J '$_X$ B=M, which is a simple module. Hence we have .B=I/=Ker./ '$_X$ B=M Also B=I '$_X$ A=D, which satisfies CM. Consequently, there are only countably many submodules Ker./. It follows that there are uncountably many 2 H with Ker./ D K=I fixed. But Hom$_X$.B=K;I=J/ is even finite since B=K is simple and I=J is finite. Thus we have reached a contradiction. □

Corollary 2. *Let* G *be a virtually hypocentral group of finite exponent. If* G *satisfies* CG*, then it is finite.*

Proof. First of all recall that a group is hypocentral if its lower central series reaches the identity subgroup when continued transfinitely. Let H G G, where H is a hypocentral group and G=H is finite. Then G=H$_0$ is finite by Theorem 4, so H D XH$_0$, where X is a finitely generated subgroup. Hence H D X ${}_i$.H/ for all positive integers i. Thus H= ${}_i$.H/ is finitely generated and hence is finite, so by the solution of the Restricted Burnside Problem it has boundedly finite order for all i.

Therefore $_i$.H/ D $_{iC1}$.H/ for some i, which implies that $_i$.H/ D 1, so H, and hence G, is finite. □

Our next aim is to determine the structure of periodic hypercentral groups with the property CG.

Theorem 5. *Let* G *be a periodic hypercentral group. Then* G *has* CG *if and only if it is an extension of a central divisible subgroup by a direct product of finite* p-*groups for distinct primes* p*.*

Proof. It suffices to prove the result when G a p-group. Since divisible subgroups of G lie in the center ([2]; see also [13, Section 9.2]), we can assume there are no non-trivial divisible subgroups. Form the lower Frattini series G D F_0 F_1 ; thus F_{iC1} D .F_i/ D $F_i^pF_{i0}$. By Theorem 4 each G=F_i is finite and G D XF_1 for some finite subgroup X, which implies that G D XF_i for all i. Since X is finite, it follows that$_p$ G=F_i has boundedly finite order and F_i D F_{iC1} for some i, that is, F_i D $^F{}_i$ F $_0$. Now a hypercentral group cannot have a non-trivial perfect factor; hence F_i D F_i^p, so that F_i is divisible and hence is trivial. Therefore G is finite. Conversely, a group with the structure in the statement has CG because the divisible subgroup is contained in every maximal subgroup. □

By way of contrast we note that McLain’s characteristically simple p-group (see [13, Section 6.2]) has no maximal subgroups and its center is trivial; of course, it is a Fitting group. A further limitation to progress in this direction is indicated by a construction of Vaughan-Lee and Wiegold [16] of countably infinite, locally nilpotent groups of finite exponent that have no maximal subgroups.

Next we discuss soluble groups satisfying chain conditions. First comes a useful result concerning the module property CM.

Lemma 4. *Let* R *be a ring and* A *a countable* R-*module. If* A *is either noetherian or artinian, then it has* CM*.*

Proof. If A is noetherian, each submodule is finitely generated and, since A is countable, it has countably many submodules. Hence A has CM. Next let A be artinian. The intersection I of all the maximal submodules of A is the intersection of finitely many of them, which implies that A=I is noetherian. By the first part, A has CM. □

Theorem 6. *A soluble group satisfying* max-n *has* CG*.*

Proof. Let G be a soluble group with derived length d satisfying max-n. Recall that G is finitely generated by a theorem of Hall [5]. The result is clearly true when d 1: we argue by induction on d > 1. Let A D $G^{.d}$ $^{1/}$. Suppose that there are uncountably many maximal subgroups M of G. Since the result holds for G=A, there are uncountably many M such that A 6 M; thus G D MA and M \ A G G.

Now A satisfies max-G, so there are countably many possibilities for M \ A. Hence

there are uncountably many maximal subgroups M with M \A D B fixed.

Consider

G=B D M=B Ë A=B

and observe that Der.G=A;A=B/ is countable because G=A is finitely generated and A=B is countable. Therefore there are countably many complements of A=B in G=B and hence countably many maximal subgroups M, a contradiction. □

Theorem 6 can also be deduced from the much more general Corollary 3 below. It is natural to ask if the dual theorem for soluble groups with min-n holds; this is a difficult problem because nothing is known about the structure of soluble groups with min-n, except for the theorem of Baer [1] that they are periodic and hence locally finite. It was shown by McDougall [10] that metabelian groups with min-n are countable, and from this it follows readily that the same holds for metanilpotent groups with min-n. On the other hand, Hartley [6] constructed examples of uncountable soluble groups of derived length 3 with min-n.

The most general result about soluble groups with min-n that we have found is the following criterion for CG.

Theorem 7. *Let* G *be a countable, periodic soluble group. Assume that there is a normal series* G D L_0 L_1 $L_{`}$ D 1*,* ` > 0*, with nilpotent factors such that* L_i=L_{iC1} *satisfies* min-$L_{i\ 1}$ *for* i D 0;1;2;:::;` 1*. Then* G *has* CG*.*

When i D 0, the condition on the factor is to be interpreted as requiring G=L_1 to satisfy min, that is, to be a Cernikov group.ˇ

Proof. If ` D 1, then G is a Cernikov group and the result is clearly true; we proˇceed by induction on ` 2. Set A D $L_{`\ 1}$ and B D $L_{`\ 2}$. Since A is nilpotent, A^0 .G/; thus we can pass to G=A_0 and assume that A is abelian. Assume the result is false; since G=A has CG by induction, there exist uncountably many maximal subgroups M not containing A; thus G D MA, M \ A G G and M \ A is a maximal G-submodule of A. Since A has min-B, it has min-G, so by Lemma 4 it satisfies CM as a G-module. Consequently, there are uncountably many maximal subgroups M with M \ A D A_0 fixed. By passing to G=A_0, we can assume that M \ A D 1 for all such M. Thus G D M Ë A and A is a simple G-module.

Next C_A.B/ D 1 or A. In the latter event B is nilpotent and there is a series of shorter length with the same properties, so the result follows by induction on `. Assume therefore that C_A.B/ D 1. By hypothesis A satisfies min-B and also B=A is nilpotent; therefore H^1.B=A;A/ D 0 (see [8, 10.3.2]). From 1 ! B=A ! G=A ! G=B ! 1 and the standard 5-term cohomology sequence we obtain 0 ! H^1.G=B;C_A.B// ! H^1.G=A;A/ ! H^1.B=A;A/. Hence H^1.G=A;A/ D 0 and Der.G=A;A/ D Inn.G=A;A/. Since A is countable, so is Der.G=A;A/ and it follows that there are countably many maximal subgroups M. □

Taking ` 2 in Theorem 7, we deduce that *a metanilpotent group satisfying* min-n *has* CG*.* When ` D 3, the statement of the theorem is that if G has min-n and L_2 has min-L_1, then G has CG.

© 2025 Computer Engineering. All Rights Reserved.

The proof of Theorem 7 shows that in order to prove that soluble groups of nilpotent length 3 satisfying min-n have CG what has to be established is this: H^1.Q;B/ is countable whenever Q is a metanilpotent group with min-n and B is a simple Q-module. In the light of this observation it is worthwhile noting the following.

Remark. Hartley's uncountable soluble groups with min-n and derived length 3 have CG.

We will briefly describe the construction. Let p;q;r be distinct primes such that p − q 1, but q j r 1. Let H be a Carinˇ .p;q/-group, i.e., the split extension of an infinite elementary abelian q-group E by a group of type p_1 acting faithfully and irreducibly on E. Hartley constructed an H-module A which is artinian and uniserial with length , the first uncountable ordinal. This means that the set of proper submodules of A is well ordered by inclusion, so its members form an ascending series of ordinal type . Then G D H Ë A is a soluble group of derived length 3 satisfying min-n. Observe that A has no maximal submodules since is a limit ordinal, so that all the maximal subgroups of G contain A. Hence G satisfies CG since G=A does.

The section ends with a characterization of metanilpotent groups with CG.

Theorem 8. *Let* G *be a metanilpotent group and let* A G G*, where* A *and* G=A *are nilpotent. Then* G *has* CG *if and only if the following conditions hold:*

(i) $G=G^pG_0$ *is finite for each prime* p.

(ii) *Each maximal* G*-submodule of* A^{ab} *has countable index in* A^{ab}*.*

(iii) *There are countably many maximal* G*-submodules of* A^{ab} *with non-*G*-trivial quotients in* A^{ab}*.*

Proof. In the first place A^0 .G/, so that G has CG if and only if $G=A_0$ does; thus we may assume that A is abelian. Assume that G satisfies CG. Then $G=G^pG_0$ has CG and hence it is finite for all primes p. Next let B denote a maximal G-submodule of A. If A=B is trivial as a G-module, jA W Bj is finite. Now assume that A=B is a non-trivial G-module. Since G=A is nilpotent and A=B is simple and non-trivial as a G-module, we have H^2.G=A;A=B/ D 0 (see [8, 10.3.2]). This shows that G=B splits over A=B, so G D X.B/A, where X.B/ G and X.B/ \ A D B. Clearly X.B/ is maximal in G, so there are only countably many subgroups X.B/. Since X.B/ \ A D B, it follows that there are countably many submodules B. In addition jG W X.B/j D jA W Bj, which implies that jA W Bj is countable. Thus conditions (i)–(iii) are necessary.

Conversely, assume that the three conditions are satisfied, but G does not have the property CG. Now G=A has CG since it is nilpotent and each G=G^pG_0A is finite. Hence there are uncountably many maximal subgroups M of G not containing A. Thus we have G D MA, M \ A G G and M \ A is a maximal G-submodule of A. If A=M \ A were G-trivial, then M G G and jG W Mj D p, a prime. Consequently, G^pG^0 M and, as G=G^pG_0 is finite, there can be only countably many such subgroups M.

From now on we assume that A=M \ A is non-trivial as a G-module. By (iii) there are countably many possibilities for M \ A. Consequently, there must exist an uncountable number of maximals M such that C D M \ A is fixed and A=C is a simple, non-trivial G-module. Therefore we have H^1.G=A;A=C/ D 0, whence Der.G=A;A=C/ D Inn.G=A;A=C/. Moreover, A=C is countable by (ii), which means that Der.G=A;A=C/ is countable. However, G=C D .M=C/ Ë .A=C/, so we reach the contradiction that there are countably many subgroups M. □

4 Modules with countably many maximal submodules

In this section we undertake a more detailed study of the module condition CM. Unlike CG this property is closed under forming extensions, for countable rings at least.

Lemma 5. *Let* R *be a countable ring and let* A *be an* R*-module with a submodule* B*. If* B *and* A=B *have* CM*, then* A *has* CM*.*

© 2025 Computer Engineering. All Rights Reserved.

Proof. Assume the result is false, so there are uncountably many maximal submodules M of A not containing B. Thus A D M C B and M \ B is a maximal submodule of B because B=M \ B 'R A=M. Since B has CM, there are uncountably many submodules M with M \B D C fixed. Then, with M fixed, we have A=C D M=C ˚ B=C and M=C 'R A=B, which has CM. Let MN ¤ M be some other maximal submodule of A containing C but not B. Then .M \ M/=CN is a maximal submodule of M=C since M=M \ MN 'R A=MN . Hence there are uncountably many submodules MN with M \ MN D D fixed. However, A=M and A=MN are simple and countable, which implies that A=D is a countable noetherian R-module. This leads to the contradiction that there are just countably many modules M=DN .

□

The next result is very simple, but it features a property that will be used to construct a large class of finitely generated groups with CG.

Lemma 6. *Let* R *be a countable ring with identity and let* A *be an* R*-module. Then the following are equivalent:*

(i) *Every finitely generated submodule of* A *has* CM*.*

(ii) *Every cyclic submodule of* A *has* CM*.*

Proof. Assume that each cyclic submodule has CM and let

$$B \; D \; a_1R \; C \; a_2R \; C \quad C \; a_nR$$

be a finitely generated submodule of A with a_i 2 A. By hypothesis each a_iR has CM and by Lemma 5 this property is extension closed; hence B has CM. The converse is obvious. □

Notice the consequence: *if* G *is a countable group and* ZG *satisfies* CI*, then every finitely generated* G*-module has* CM. Next comes a key lemma involving the property characterized in Lemma 6.

Lemma 7. *Let* G *be a finitely generated group with a normal subgroup* A*. Assume that* A *is either finite or abelian with every finitely generated* G*-submodule satisfying* CM*. Then all but a countable number of maximal subgroups of* G *contain* A*.*

Proof. Assume the result is false. Then A must be infinite by Theorem 2, so it is abelian. By assumption there are uncountably many maximal subgroups of G not containing A; let M be one of them. Choose a 2 MnA and set A.M/ D a^G. Then M \ A.M/ G MA.M/ D G and also M \ A.M/ is a maximal G-submodule of A.M/. There are only countably many possibilities for A.M/ since it is a cyclic G-module and A is countable. By hypothesis A.M/ satisfies CM, so there are only countably many possible submodules M \ A.M/. Consequently, there is an uncountable set M of maximal subgroups M of G such that A.M/ D B and M \ A.M/ D C with B and C fixed for all M 2 M.

Let M 2 M and consider the group G=C D .M=C/ Ë .B=C/. As G is finitely generated and B=C is countable, Der.G=B;B=C/ is countable, showing that there are only countably many complements of B=C in G=C. Hence there are countably many M 2 M, a contradiction. □

We are now in a position to produce a large class of finitely generated groups with CM.

Theorem 9. *Let* G *be a finitely generated group with a normal subgroup* N*. Assume that* N *has an ascending* G*-invariant series in which each infinite factor is abelian with the property that all its finitely generated* G*-submodules satisfy* CM*. Then all but a countable number of maximal subgroups of* G *contain* N*.*

Proof. Let ¹N¸ j ¸ ˇº be the given ascending G-invariant series in N. If the result is false, there is a least ordinal ¸ for which there is an uncountable set M of maximal subgroups of G none of which contains N¸.

Suppose first that ¸ is not a limit ordinal. Then all but a countable number of subgroups in M contain N¸ 1. By hypothesis either N¸=N¸ 1 is finite or it is abelian and its finitely generated G-submodules satisfy CM. Thus we can apply Lemma 7 to the group G=N¸ 1, concluding that all but a countable number of subgroups in M contain N¸, a contradiction that shows ¸ to be a limit ordinal. Let < ¸; then all but a countable number of subgroups in M contain N . Keeping in mind that N is countable, so that there are countably many such subgroups N , we conclude that all but a countable number of subgroups in M contain every N for < ¿, and hence contain N¿, a final contradiction. □

Of course, if in addition G=N has CM, the same is true of G.

Corollary 3. *Let* G *be finitely generated group with a normal subgroup* N*. Assume that* N *has an ascending* G*-invariant series in which each infinite factor is abelian and its finitely generated* G*-submodules satisfy* max-G *or* min-G*. Then all but a countable number of maximal subgroups of* G *contain* N*.*

Proof. This follows at once from Theorem 9 and Lemma 4. □

From Corollary 3 one can read off many types of finitely generated CG-groups. For example, N could be hypercentral in G or even hypercyclically embedded in G and G=N nilpotent-by-polycyclic or soluble with finite rank.

Finally, here is another simple result which shows connections between the properties CG, CM and CI.

Proposition 1. *Let* Q *be a finitely generated group with* CG*. Then the following statements are equivalent:*

(i) *Every extension of a finitely generated* Q*-module by* Q *has* CG*.*

(ii) *Every finitely generated* Q*-module has* CM*.*

(iii) *The group ring* ZQ *has* CI*.*

Proof. (i)) (ii). Let A be a finitely generated Q-module and write G D Q Ë A, the natural semidirect product. By hypothesis G has CG. If M is a maximal submodule of A, then QM is a maximal subgroup of G, so there are countably many subgroups QM. This implies that there are countably many submodules M because M D A \ QM.

(ii)) (i). Let G be an extension of a finitely generated Q-module A by Q. By hypothesis every finitely generated submodule of A has CM. Since G is finitely generated, Lemma 7 shows that G has CG.

(ii)) (iii). This is obvious.

(iii)) (ii). As has already been observed, this follows from Lemma 6. □

5 Group rings with countably many maximal right ideals

We move on to study the property CI for a group ring ZG. This is a very strong property since it is preserved on passing to subgroups of G. The basis for this is:

Lemma 8. *Let* R *be a ring with identity and let* G *be a group with a subgroup* H*. If* RG *has* CI*, the same is true of* RH*.*

This follows from a standard result about group rings: *if* H *is a subgroup of a group* G *and* I *is a maximal right ideal of* RH*, there is a maximal right ideal* J *of* RG *such that* I D J \ ZH – for this see [12, Lemma 6.1.2].

The next result may be compared with Theorem 1.

Proposition 2. *Let* G *be a countable group with a subgroup* H *of finite index. Then* ZG *has* CI *if and only if* ZH *has* CI*.*

Proof. In the first place Lemma 8 shows that ZH will have CI if ZG does. Assume that ZH has CI. By Lemma 8 again we can replace H by H_G, so we may as well assume that H G G. Let T be a transversal to H in G; then

ZG D $L_{t2}{}^{T}$H.Z-H/tmodules haveand consequentlyCM, we conclude from Lemma 6

thatZG is finitely generated as an H-Zmodule.G has

Since cyclic countably many maximal H-

submodules.

Let M be a maximal right ideal of ZG. Now M is contained in a maximal H-submodule L of ZG, since the latter is finitely generated as an H-module. If t 2 T , then Lt is also a maximal H-submodule of ZG containing M. Hence

M TD^{t}2^{T}LNLt. ThereforeD LN andMLN is the intersection of finitely many maximalis a proper right ideal of ZG, from which it followsH-sub-

that M

modules of ZG. Since ZG has countably many maximal H-submodules, there are only countably many possibilities for M. □

Next we examine the effect on the structure of a soluble group of imposing the property CI on its group ring.

© 2025 Computer Engineering. All Rights Reserved.

Theorem 10. *Let* G *be a soluble group such that* ZG *has the property* CI*. Then* G *is a minimax group with no sections of type* p^1 p_1.

As already noted, the groups appearing in Theorem 10 are exactly the soluble groups that have countably many subgroups [4]. In the proof of the theorem we will use the following lemma.

Lemma 9. *Let* A *be an abelian group which has an uncountable set of subgroups* ^{1}B j 2 fº *such that* A=B *is a periodic, locally cyclic* p_0*-group for some prime* p*. Then* ZA *has uncountably many maximal ideals.*

Proof. It follows from the hypothesis that A=B is isomorphic with a subgroup of the multiplicative group of ZN_p, the algebraic closure of Z_p; moreover, the image of A=B generates a subfield F of ZN_p as a ring. It follows that F is a simple A-module via the natural action of A=B. Hence F $'_A$ ZA=M, where M is a maximal ideal of ZA. The kernel of the action of A on F is clearly B. Therefore B ¤ B implies that M ¤ M and consequently ZA has uncountably many maximal ideals. □

Proof of Theorem 10*.* Since the property CI is inherited by group rings of subgroups and quotients, we may assume that G is abelian. Furthermore, it is sufficient to prove that G cannot be an infinite elementary abelian p-group, an infinite direct product of groups of distinct prime orders or a group of type p^1 p_1. If G is one of these groups, it is easily verified that there is a set of subgroups with the properties listed in Lemma 9. Therefore ZG has uncountably many maximal ideals in ZG, and the theorem is established. □

Whether the converse of Theorem 10 is true remains open, but we are able to prove it in the case of abelian-by-finite groups.

Theorem 11. *Let* G *be a finite extension of an abelian minimax group without factors of type* p^1 p_1 *for any prime* p*. Then* ZG *satisfies* CI.

Proof. By Proposition 2 we may assume G to be abelian and minimax. Let M be a fixed maximal ideal of ZG and write F D ZG=M, which is a simple G-module and also a field. We show that Q D G=C_G.F/ is locally (finite cyclic).

There is a monomorphism from Q into F , the multiplicative group of F, whose image generates F as a ring; we regard Q as a subgroup of F . Since Q is minimax, it has a free abelian subgroup A such that Q=A is periodic. Let R D RghAi, the subring of F generated by A. Since each element of Q has a positive power in A and F D RghQi, it follows that F is integral over R. But F is a field, so R is also a field. Since R is finitely generated as a ring, it is finite and therefore A D 1, showing that Q is periodic. In addition Q D G=C_G.F/ acts on F as an irreducible group of automorphisms, so we can apply a theorem of Baer ([1]; see also [13, Lemma 5.26]) to show that F has characteristic p > 0 and hence Q is a locally cyclic p_0-group.

Assume that ZG has uncountably many maximal ideals M. Clearly we may suppose that I_G 6 M. There are only countably many subgroups C_G.ZG=M/, from which we deduce that there are uncountably many maximal ideals M with C_G.ZG=M/ D C fixed and ZG=M elementary abelian p for some p. Since IN_C M, we can pass to ZG=IN_C ' Z.G=C/ and assume that C D 1. Thus G is a locally cyclic p_0-group.

We know that G acts faithfully on ZG=M and p does not belong to the spectrum of G; also of course M is a prime ideal. We are therefore in a position to apply a result of Segal [14, Corollary 1.2] to conclude that M is finitely generated as an ideal of ZG. However, this means that there only countably many maximal ideals M. □

An immediate application of Theorems 10 and 11 is a characterization of locally finite groups whose group rings satisfy CI.

Corollary 4. *Let* G *be a locally finite group. Then* ZG *satisfies* CI *if and only if* G *is a Cernikov group whose maximum divisible subgroup is locally cyclic.*ˇ

Proof. Assume that ZG has CI. By Lemma 8 and Theorem 10 abelian subgroups of G are Cernikov groups. The well-known theorem of Šunkov [15] shows thatˇ G is a Cernikov group. It follows from Theorem 10 that the maximum divisible sub-ˇ group of G is locally cyclic. The converse is true by Theorem 11. □

6 Soluble groups with uncountably many maximal subgroups

One approach to constructing soluble groups that do not have CG is to look for group rings which do not satisfy CI. Let Q be a group such that ZQ does not have CI. Then the natural semidirect product G D Q ËZQ ' ZwrQ has uncountably many maximal subgroups of the form QM, where M is a maximal right ideal of ZQ. If Q is soluble or finitely generated, then so is G.

Examples. (i) Let Q be a group of type p^1 p_1, where p is a prime. Then ZQ does not have CI by Theorem 10, so G D ZwrQ is a countable metabelian group that does not have CG.

(ii) Let q be a prime and put Q D Z_q wrZ. Since the base group of the wreath product is infinite elementary abelian, ZQ does not have CI. Hence G D ZwrQ is a finitely generated soluble group of derived length 3 which does not have CG. This is the minimum derived length possible here since finitely generated metabelian groups satisfy CG.

(iii) Let H D hx;a j a^x D a^pi, where p is a prime, and put Q D H H. Because Q has a quotient of type p^1 p_1, its group ring does not have CI. Consequently, G D ZwrQ is a finitely generated, torsion-free soluble group of derived length 3 that does not satisfy CG.

(iv) Let A be a non-trivial countable abelian group. By a well-known construction of Hall ([5]; see also [8, 4.1.3]), there is a finitely generated center-bymetabelian group Q whose center is isomorphic with A. If we choose A so that ZA does not have CI, then neither will ZQ, and the group ZwrQ will fail to have CG.

(v) The final example is an explicit construction of uncountably many maximalright ideals in a group algebra in case (ii). The idea is due to de Cornulier

[3]. Since the construction has not appeared in print form, we present an account of a generalized version.

Proposition 3. *Let* p;q *be primes such that* p 1 mod q *and let* Q D Z_q wr Z*.*

Then Z_pQ*, and hence* ZQ*, does not satisfy* CI*.*

Proof. Write Q D hu$_0$i wr hti, where u^q_0 D 1. Let M be a Z_p-vector space with a countably infinite basis ¹e$_i$ j i 2 Zº. Since q divides p 1, there is a primitive qth root of unity ! in Z_p. Denote by F the set of all functions from Z to h!i. Each choice of from F turns M into a right Q-module M by means of the rules

.e$_i$/u$_0$ D .i/e$_i$;   .e$_i$/t D e$_{iC1}$ .i 2 Z/:

Writing u$_k$ D $u^{t}_{0}{}^{k}$, we have .e$_i$/u$_k$ D .i k/e$_i$. Regarding F as the set of all bi-infinite sequences of elements from h!i, we define F$_0$ to be the subset of all 2 F such that every finite sequence of elements of h!i occurs as a (consecutive) subsequence of . Notice that F$_0$ is uncountable.

We show next that M is a simple Q-module if 2 F$_0$. To prove this let

0assumption the sequence¤ m 2 M and write m!D;!;:::;!;P`iDk aie1i, with, where`
k k entries` and a!i , occurs as a sub-2 Z$_p$, a` ¤ 0. By

sequence of . Hence there is a conjugate uN of u$_0$ such that .e$_i$/uN D !e$_i$ for i D k;k C 1;:::;` 1 and .e`/uN D e`. Therefore the submodule S D .m/ZQ contains the element

q 1 ! ` 1 q 1 ! m X uN$_j$ D X a$_i$ X !$_j$ e$_i$ C qa`e` D qa`e`:

jD0 iDk jD0

Hence qa`e` 2 S and e` 2 S. It follows that S contains all the e$_i$, so S D M and M is a simple Q-module.

© 2025 Computer Engineering. All Rights Reserved.

Next define a Q-module homomorphism W Z_pQ ! M by .1/ D e_0. Since M is simple, is surjective and M $'_Q$ZQ=K, where K D Ker./, which is a maximal right ideal of ZQ. Suppose that K D K_N, where ;N 2 F_0. For any j we have

.u_j/ D ..1//u_j D .e_0/u_j D . j/e_0:

Hence .u_j b/ D .. j/ b/e_0 for any b 2 Z_p. Thus u_j b 2 K if and only if . j/ D b. Suppose that ¤ N, so that . j/ D b ¤ bN D .N j/ for some j 2 Z. Hence u_j b 2 K D K_N, so that .N j/ D b and b D bN. By this contradiction D N. It follows that the map 7! K is injective, which shows that Z_pQ has uncountably many maximal right ideals K, 2 F_0. □

Bibliography

[1] R. Baer, Irreducible groups of automorphisms of abelian groups, *Pacific J. Math.* 14 (1964), 385–406.

[2] S.N. Cernikov, On the theory of complete groups (in Russian),ˇ *Mat. Sb. (N.S.)* 22(64) (1948), 319–348; translation in *Amer. Math. Soc. Transl. Ser. 1* 56 (1951), 3–49.

[3] Y. de Cornulier, MathOverflow, 2016, mathoverflow.net/questions/249071/.

[4] G. Cutolo and H. Smith, Groups with countably many subgroups, *J. Algebra* 448 (2016), 399–417.

[5] P. Hall, Finiteness conditions for soluble groups, *Proc. Lond. Math. Soc. (3)* 4 (1954), 419–436.

[6] B. Hartley, Uncountable Artinian modules and uncountable soluble groups satisfying Min-n, *Proc. Lond. Math. Soc. (3)* 35 (1977), no. 1, 55–75.

[7] B. Hartley and D. McDougall, Injective modules and soluble groups satisfying the minimal condition for normal subgroups, *Bull. Aust. Math. Soc.* 4 (1971), 113–135.

[8] J.C. Lennox and D.J.S. Robinson, *The Theory of Infinite Soluble Groups*, Oxford Math. Monogr., Clarendon Press, Oxford, 2004.

[9] G.A. Margulis and G.A. Soˇıfer, Maximal subgroups of infinite index in finitely generated linear groups, *J. Algebra* 69 (1981), no. 1, 1–23.

[10] D. McDougall, Soluble groups with the minimum condition for normal subgroups, *Math. Z.* 118 (1970), 157–167.

[11] A.Y. Ol’shanskiˇı, *Geometry of Defining Relations in Groups*, Math. Appl. (Soviet Ser.) 70, Kluwer Academic Publishers, Dordrecht, 1991.

[12] D.S. Passman, *The Algebraic Structure of Group Rings*, Pure Appl. Math., WileyInterscience, New York, 1977.

[13] D.J.S. Robinson, *Finiteness Conditions and Generalized Soluble Groups. Part 1*, Ergeb. Math. Grenzgeb. (3) 62, Springer, Berlin, 1972.

[14] D. Segal, On the group rings of abelian minimax groups, *J. Algebra* 237 (2001), no. 1, 64–94.

[15] V.P. Šunkov, Locally finite groups with a minimality condition for abelian subgroups, *Algebra Logika* 9 (1970), 579–615.

[16] M.R. Vaughan-Lee and J. Wiegold, Countable locally nilpotent groups of finite exponent without maximal subgroups, *Bull. Lond. Math. Soc.* 13 (1981), no. 1, 45–46.